ACHIRALITY OF SOL 3-MANIFOLDS, STEVENHAGEN CONJECTURE AND SHIMIZU'S L-SERIES

YE TIAN, SHICHENG WANG, AND ZHONGZI WANG

Contents

1. Introduction 1
2. Commensurable Classes of Sol 3-manifolds and Their Discriminants 2
3. Achirality, Gauss' Theory of Class Groups and Genus 7
4. A Density Result 11
5. Shimizu L-functions 13
References 15

1. Introduction

In this paper, we assume that all manifolds M are closed.

A closed orientable manifold is called *achiral*, if it admits an orientation reversing homeomorphism. We call two manifolds M_1 and M_2 commensurable, if they have a common finite cover. A commensurable class of closed manifolds is called achiral if it contains an achiral element. Note that a commensurable class containing a non-orientable element must be achiral. The achirality is fundamental in topology for classification of oriented manifolds, specially in dimension 3, where the concept achirality is originated.

Recall that Thurston's eight geometries among 3-manifolds are [Th] [Sc],

$$S^3, \quad \mathbb{E}^3, \quad \mathbb{H}^2\times\mathbb{E}^1, \quad S^2\times\mathbb{E}^1, \quad \text{Nil}, \quad \widetilde{\text{PSL}(2,\mathbb{R})}, \quad \mathbb{H}^3, \quad \text{Sol}.$$

Here $\mathbb{H}^n, \mathbb{E}^n, S^n$ indicate the n-dimensional hyperbolic, Euclidean, and spherical geometries, respectively. It is known that 3-manifolds supporting one of the first six geometries form only one commensurable class, and commensurable classes are achiral for the manifolds supporting first four geometries, and non-achiral for manifolds supporting next two geometries. On the other hand 3-manifolds supporting one of the last two geometries form infinitely many commensurable classes, and the achirality of 3-manifolds supporting Sol and hyperbolic geometries, as well as their commensurable classes are complicated.

We discuss achirality of orientable Sol 3-manifolds and their commensurable classes in this paper.

Each commensurable class $\mathcal{M}$ of Sol 3-manifold contains an orientable torus bundle $M = M_\phi$, eigenvalues of whose monodromy map ϕ generates over $\mathbb{Q}$ a real quadratic field. The discriminant of the real quadratic field is the complete topological invariant of $\mathcal{M}$, denoted by $D_\mathcal{M}$ and called the discriminant of $\mathcal{M}$. One main result, Theorem 4.4, in the paper is that among all achiral

Date: July 8, 2023.
2010 *Mathematics Subject Classification.* 55M25.
Key words and phrases. Class groups, Pell equations, Achirality of Sol 3-Manifolds.

commensurable classes of Sol 3-manifolds ordered by discriminants, the density of classes containing non-orientable elements:

$$\lim_{X\to\infty} \frac{\#\{D < X \mid \mathcal{M}_D \text{ contains an non-orientable element}\}}{\#\{D < X \mid \mathcal{M}_D \text{ is achiral}\}} = 1 - \rho,$$

where

$$\rho := \prod_{j=1}^{\infty} \left(1 + 2^{-j}\right)^{-1} = 0.41942\cdots.$$

For each orientable Sol torus bundle $M = M_\phi$, we introduce the discriminant D_M of M to be the discriminant of the fixed points of its monodromy ϕ. The integer D_M is a topological invariant of M so that $\mathbb{Q}(\sqrt{D_M})$ is the real quadratic field of the commensurable class containing M. Another result, Theorem 3.5, in this paper is that for any given $D \equiv 0, 1 \bmod 4$ non-square positive integer, the following conditions are equivalent

- there exists an achiral Sol torus bundle M with discriminant D;
- no prime factor of D is $\equiv 3 \bmod 4$ and $16 \nmid D$.

As consequences, among Sol 3-manifolds, a commensurable class $\mathcal{M}$ is achiral if and only $D_{\mathcal{M}}$ contains no prime $\equiv 3 \bmod 4$; each achiral commensurable class contains non-achiral manifolds; there are infinitely many achiral commensurable classes, however their density among all commensurable class is zero ordered by discriminants; and $\mathcal{M}$ contains an non-orientable element if and only if it contains an achiral torus semi-bundle; Corollaries 3.6, 3.7, 4.2.

In his study on cusp cross-sections of Hilbert modular varieties, Hirzebruch made a conjecture relates the signature defects of an orientable Sol torus bundle to the special value of its corresponding Shimizu's L-function. In this paper, we also show that, Theorem 5.1, for an oriented Sol torus bundle M, the following are equivalent:

1) M is achiral;
2) the Shimizu's L-series of M is identically zero.

If monodormies of two oriented Sol torus bundles are negative to each other, they are not homeomorphic but their Shimuz L-series are the same. We have that non-achiral oriented Sol tours bundles of 3 dimensional are determined by their Shimizu L-functions up to such a relation.

Theorem 4.4 follows from the positive answer of Stevenhagen Conjecture, Theorem 3.5, and Corollary 4.1 which claims that a commensurable class contains a non-orientable element if and only if the corresponding negative Pell equation has a solution. Both the proofs of Theorem 3.5 and 5.1 are based on Theorem 3.4 which claims that a Sol 3-manifold M_ϕ is achiral if and only if $[-Q_\phi] = \pm[Q_\phi]$ in class group $C(D_M)$ for its corresponding quadratic form Q_ϕ. The proof of Theorem 3.5 also highly relies on Gauss' genus theory.

Remark 1.1. Call a closed 3-manifold M **virtually achiral** if it has an achiral finite cover. A commensurable class $\mathcal{M}$ is achiral is equivalent to that each manifold in $\mathcal{M}$ is virtually achiral (Lemma 2.9). We can translate Corollaries 3.6, 3.7 as: A Sol 3-manifold is virtually achiral if and only the discriminant of its commensurable class contains no prime $\equiv 3 \bmod 4$, and to have a achiral finite cover are rare among Sol 3-manifolds. The study of various virtual properties, say virtually Haken, virtually positive volume, virtual dominations, are important and active topics, see a survey [LS].

2. Commensurable Classes of Sol 3-manifolds and Their Discriminants

Classification of Sol 3-manifolds. Recall the Sol geometry is the Lie group $\mathbb{R}^2 \rtimes \mathbb{R}^1$ with structure given by the representation $\mathbb{R}^1 \to \mathrm{GL}_2(\mathbb{R})$, $z \mapsto \begin{pmatrix} e^z & \\ & e^{-z} \end{pmatrix}$, together the invariant Riemann metric

$e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$. A connected closed manifold is called Sol 3-manifold if it is modelled on Sol, i.e. is of form Sol/Γ for some discrete subgroup Γ of Isom(Sol). There are two types Sol 3-manifolds:

- let $T = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ be a torus, $I = [0,1]$, and $\phi \in \mathrm{GL}_2(\mathbb{Z})$ Anosov matrix, i.e. its eigenvalues are real but not $\{\pm 1\}$. Define

$$M_\phi = T \times I/((x,y)\phi, 0) \sim (x,y,1).$$

  Thus M_ϕ is a tours bundle over a circle.
- let

$$N = T \times I/(x,y,z) \sim (x+\pi, -y, 1-z)$$

  a twisted I-bundle of the Klein bottle $K = T/(x,y) \sim (x+\pi, -y)$. For each $\psi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $abcd \neq 0$, define the semi-torus bundle by gluing two N's along their boundary $T \times \{1\}$ via ψ:

$$N_\psi = N \cup N/(x,y,1) \sim ((x,y)\psi, 1),$$

The following classification result of Sol 3-manifolds is basically known except the non-orientable case needs some explanations.

Proposition 2.1. *Any Sol 3-manifold is homeomorphic to one of the following*

(1) *Torus bundles M_ϕ: two M_ϕ and $M_{\phi'}$ are homeomorphic if and only if ϕ' is* $\mathrm{GL}_2(\mathbb{Z})$*-conjugate to $\phi^{\pm 1}$. Moreover, M_ϕ is orientable if and only if $\phi \in \mathrm{SL}_2(\mathbb{Z})$.*

(2) *Torus semi-bundles N_ψ: two N_ψ and $N_{\psi'}$ are homeomorphic if and only if*

$$\psi' = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \psi^{\pm 1} \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

Moreover, N_ψ is always orientable.

Proof. Orientable Sol 3-manifolds are classified by ([Ha], Theorems 2.6 and 2.8, see also [SWW] Theorems 2.3 and 2.4). Now let M be a non-orientable Sol 3-manifold. Then the orientable double cover of M must be either a torus bundle or torus semi-bundle. In the first case, M must also be a torus bundle by the classification of free involutions on torus bundles (see [Sak]). In fact, it is exactly the Case (II) in [Sak] p 168. In the second case, M is given by a free involution on torus semi-bundle which must be orientable by [Theorem 2] in [BGV]. □

Fix orientations on T and I, each orientable torus bundle M_ϕ, $\phi \in \mathrm{SL}_2(\mathbb{Z})$, inherits an orientation, and also denote by M_ϕ the oriented torus bundle.

Lemma 2.2. *For each oriented M_ϕ, $M_{\phi^{-1}} = -M_\phi$. where $-M$ is the M with opposite orientation.*

Proof. By definition we have

$$M_{\phi^{-1}} = \frac{T \times [0,1]}{(x,1) \sim (\phi^{-1}(x), 0)}.$$

Note identify the top to the bottom via ϕ^{-1} is the same as identify the bottom to top via ϕ, and later is homeomorphic to M_ϕ via an orientation reversing homeomorphism which is obtained by changing the orientation of $[0,1]$. We proved the lemma. □

Lemma 2.3. *Two oriented tours bundles M_ϕ and $M_{\phi'}$ are homeomorphic if and only if ϕ' is* $\mathrm{SL}_2(\mathbb{Z})$*-conjugate to ϕ or $w\phi^{-1}w$, where $w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$.*

Proof. By Proposition 2.1 (1), $M_{\phi'}$ is homeomorphic to M_ϕ if and only if $\phi' = A\phi^{\pm}A^{-1}$, $A \in \mathrm{GL}_2(\mathbb{Z})$. If $A \notin SL_2(\mathbb{Z})$, we write $A = Bw$ with $B \in SL_2(\mathbb{Z})$. So $M_{\phi'}$ is homeomorphic to M_ϕ if and only if ϕ' is $\mathrm{SL}_2(\mathbb{Z})$ conjugate to $\phi^{\pm 1}$ or $w\phi^{\pm 1}w$. The conclusion follows by Lemma 2.2 and that $\det w = -1$, that is w is orientation reversing on T. □

Commensurable classes of Sol 3-manifolds. We call two manifolds M_1 and M_2 commensurable, if they have a common finite cover. It is easy to see that commensurable relation is an equivalent relation. We will use $\mathcal{M}$ to denote the commensurable class of M.

Suppose $K = \mathbb{Q}(\sqrt{d})$ is a real quadratic field, where $d > 0$ is square free. Define the fundamental discriminant D_K of K by $D_K = d$ if $d \equiv 1 \bmod 4$ and $D_K = 4d$ otherwise [EW, p.92].

Each commensurable class $\mathcal{M}$ of Sol 3-manifold contains an orientable torus bundle, eigenvalues of whose monodromy map generates over $\mathbb{Q}$ a real quadratic field K. The fundamental discriminant D_K, denoted by $D_{\mathcal{M}}$ and called the discriminant of $\mathcal{M}$.

Proposition 2.4. *There is a one-to-one bijection between*

- *commensurable classes $\mathcal{M}$ of Sol 3-manifolds;*
- *real quadratic fields $\mathbb{Q}(\sqrt{D_{\mathcal{M}}})$ (or equivalently, fundamental positive discriminants $D_{\mathcal{M}}$),*

For each torus bundle $M_\phi : T \to M_\phi \to S^1$, we have a short fiber exact sequence

$$1 \to \pi_1(T) \to \pi_1(M_\phi) \to \pi_1(S^1) \to 1. \qquad (2.1)$$

For each covering map $p : T \to T$, we can homotopy p to be non-degenerated linear map, that is $p \in M_2(\mathbb{Z})$ and p is of rank 2. For each fiber preserving covering map $f : M_\psi \to M_\phi$ between torus bundles, there is an induced commutative diagram between of these exact sequences.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(T) & \xrightarrow{i'_*} & \pi_1(M_\psi) & \xrightarrow{q'_*} & \pi_1(S^1) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle f|_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \bar{f}_*} & & \\ 1 & \longrightarrow & \pi_1(T) & \xrightarrow{i_*} & \pi_1(M_\phi) & \xrightarrow{q_*} & \pi_1(S^1) & \longrightarrow & 1 \end{array} \qquad (2.2)$$

Call the covering f is vertical, if the left side vertical map $f|_*$ is an isomorphism, that is to say the covering is from the circle direction; and the covering is horizontal, if the right side vertical map $\bar{f}_*$ is an isomorphism, that is to say the covering is from the torus direction.

Call a closed orientable 3-manifold M Haken, if each embedded 2-sphere in M bounds a 3-ball in M, and there exists a closed orientable embedded surface of genus ≥ 1 in M which induce an injection π_1. Each Sol 3-manifold is Haken.

Lemma 2.5. *Suppose M_ϕ and M_ψ are orientable torus bundle supporting Sol geometry. Then*
(1) Each Sol M_ϕ has unique torus bundle structure up to isotopy.
(2) Any covering $f : M_\psi \to M_\phi$ is fiber preserving up to isotopy.

Proof. (1) Suppose T' is a torus fiber of a fiberation of M_ϕ. According to (2.1), $T \to M_\phi$ induces an injection on π_1, hence T' is incompressible. Since $|\mathrm{tr}(\phi)| > 2$, ϕ is not conjugate to $\begin{pmatrix} 1 & 1 \\ n & 1 \end{pmatrix}$, by [Ha, Lemma 5.2], T' is isotopy to T, the torus fiber of the fiberation determined by ϕ.

(2) If we fiber M_ψ with lifted torus fiberation of M_ϕ, then the covering is fiber preserving. Since M_ψ also supports Sol geometry, by (1) its original torus fiberation is isotopic to the lifted torus fiberation, and (2) follows. □

Lemma 2.6. *For each fiber preserving covering map $f : M_\psi \to M_\phi$ between torus bundles, $f = f_v \circ f_h$, where f_h is a horizontal covering and f_v is a vertical covering. Moreover*

(1) *there is a horizontal covering* $f : M_\psi \to M_\phi$ *if and only if* ψ *is* $\mathrm{GL}_2(\mathbb{Q})$*-conjugate to* $\phi^{\pm 1}$*, or equivalently,* $\mathrm{tr}\phi = \mathrm{tr}\psi$
(2) *there is a vertical covering* $f : M_\psi \to M_\phi$ *of degree* n *if and only if* ψ *is* $\mathrm{GL}_2(\mathbb{Z})$*-conjugate to* $\phi^{\pm n}$*.*

It follows that any fiber preserving covering $f : M_\psi \to M_\phi$ *has that* ψ *is* $\mathrm{GL}_2(\mathbb{Q})$*-conjugate to* ϕ^n *for some integer* $n \neq 0$*.*

Proof. Let $G = (q^*)^{-1}\bar{f}_*(\pi_1(S^1))$, G. By (2.2), G is subgroup of $\pi_1(M_\phi)$ and $f_*(\pi_1(M_\psi))$ is a subgroup of G. Since $f : M_\psi \to M_\phi$ is a finite covering, we have that G is a finite index subgroup of $\pi_1(M_\phi)$. By covering space theory, there is a 3-manifold M with $\pi_1(M) = G$ and the finite covering f is docomposed into the finite covering $f_h : M_\psi \to M$ and $f_v : M \to M_\phi$. If we lift the torus bundle structure of M_ϕ to M, then both f_h and f_v become a fiber preserving covering. One can verify that f_h is horizontal and f_v is vertical.

We now show (1). Let T be a fiber torus of M_ϕ. Since the fiber preserving covering f is horizontal, $\tilde{T} = f^{-1}(T)$ is a fiber torus of M_ψ. Cutting M_ϕ along T and M_ψ along $\tilde{T}$, we get an induced a fiber preserving covering map

$$\bar{f} : \tilde{T} \times [0,1] \longrightarrow T \times [0,1],$$

where $f(x,t) = (p(x), \epsilon t)$, where $p : \tilde{T} \to T$ is a covering map, and $\epsilon = \pm 1$ which depends on the map preserving or reversing the orientation of $[0,1]$. Then from the constructions of M_ψ and M_ϕ, we have the following commutative diagram

$$\begin{array}{ccc} \tilde{T} \times \{1\} & \xrightarrow{p} & T \times \{1\} \\ \downarrow \psi & & \downarrow \phi^\epsilon \\ \tilde{T} \times \{0\} & \xrightarrow{p} & T \times \{0\} \end{array} .$$

That is to say for any $(x,1) \in \tilde{T} \times \{1\}$, we have $\phi^\epsilon \circ p(x,1) = p \circ \psi(x,1)$, where we define $p(x,1) = (p(x),1)$. That is $(\phi^\epsilon \circ p(x),0) = (p \circ \psi(x),0)$. So we have $\phi^\epsilon \circ p = p \circ \psi$, or

$$A \circ \psi = \phi^\epsilon \circ A, \qquad (2.3)$$

where A is the rank 2 element in $M_2(\mathbb{Z})$ induced by p. Another direction is a direct construction.

It is clear that $\psi = A^{-1} \circ \phi^{\pm 1} \circ A$ implies that $\mathrm{tr}(\phi) = \mathrm{tr}(\psi)$. On the other hand, if $\mathrm{tr}(\phi) = \mathrm{tr}(\psi)$, then ϕ and ψ are conjugate in $\mathrm{GL}_2(\mathbb{R})$ and thus in $\mathrm{GL}_2(\mathbb{Q})$ and in $\mathrm{GL}_2(\mathbb{Z})$ (the equation $p\phi = \psi p$ in p is linear). In fact, let $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\psi = \begin{pmatrix} a+u & w \\ v & d-u \end{pmatrix}$, then $cv \neq 0$ and one may take $p = \begin{pmatrix} 1 & u/c \\ 0 & v/c \end{pmatrix}$.

We now show (2). First M_{ϕ^n} has a vertical covering structure of M_ϕ of degree n. On the other hand, $H_1(M_\phi, \mathbb{Z}) = \mathbb{Z} \oplus \frac{H_1(T,\mathbb{Z})}{\mathrm{Im}(\phi_* - \mathrm{Id})}$ and a vertical covering $M_\psi \to M_\phi$ of degree n corresponds to the index n subgroup of $H_1(M_\phi, \mathbb{Z})$ given by the $\mathbb{Z}$-component. By Lemma 2.5 (1), the uniqueness of fiberation, ψ is conjugate to $\phi^{\pm n}$. ∎

Now we recall some facts about real quadratic field $K = \mathbb{Q}(\sqrt{d})$ [EW, Chapters 3 and 4]which will be used in the proof of Proposition 2.4 and thereafter. For each $x = a + b\sqrt{d} \in F$, define its norm $N(x) = a^2 - b^2 d$. Call x absolutely positive if both x and $N(x)$ are positive.

Let O_K be the ring consists of all algebraic integers of K, or equivalently, the $x \in K$ whose minimal polynomial hase form $x^2 + bx + c$ for $b, c \in \mathbb{Z}$.

Call a subring $\mathcal{O} \subset O_K$ of rank 2 an order of K. It is known that $\mathcal{O} = \mathbb{Z} \oplus c^2 O_K$ for $c \in \mathbb{Z}$. Define the discriminant of $\mathcal{O}$ to be $c^2 D_F$. Note O_K itself is the maximum order of K. Let $\mathcal{O}^\times$ be group of all units of $\mathcal{O}$. Then Dirichlet Unit Theorem implies $\mathcal{O}^\times = \mathbb{Z} \oplus \mathbb{Z}_2$, and call a positive generator of $\mathbb{Z}$ a fundamental unit of $\mathcal{O}$.

Proof of Proposition 2.4. Suppose M_{ϕ_1}, M_{ϕ_2} are commensurable. Then there are two coverings $M_\phi \to M_{\phi_1}, M_{\phi_2}$ between Sol torus bundles. By Lemma 2.5, we may assume that they are fiber preserving coverings. By Lemma 2.6, we have

$$\phi = A_i \phi_i^{n_i} A_i^{-1}, \qquad i = 1, 2.$$

Here $A_i \in \mathrm{GL}_2(\mathbb{Q})$ and $n_i \neq 0$ integers. It follows that

$$\phi_2^{n_2} = (A_2^{-1}A_1)\phi_1^{n_1}(A_1^{-1}A_2).$$

Therefore, ϕ_1, ϕ_2 give rise to the same real quadratic fields.

On the other hand, suppose ϕ_1, ϕ_2 give rise to the same real quadratic field K, and let λ_1, λ_2 be their eigenvalues, respectively. As the roots of characteristic polynomial of element in $SL_2(\mathbb{Z})$, both λ_1, λ_2 are units in O_F. Let ϵ be a fundamental unit of O_K. Since $O_K^\times = \mathbb{Z} \oplus \mathbb{Z}_2$, we have

$$\lambda_1 = \pm\epsilon^{n_1}, \quad \lambda_2 = \pm\epsilon^{n_2}$$

for some integers $n_1, n_2 \neq 0$. This implies $\lambda_1^{2n_2} = \lambda_2^{2n_1}$, and therefore $\mathrm{tr}(\phi_1^{2n_2}) = \mathrm{tr}(\phi_2^{2n_1})$. By Lemma 2.6 (1), there exists a covering $M_{\phi_1^{2n_2}} \to M_{\phi_2^{2n_1}}$. Since $M_{\phi_2^{2n_1}}$ covers M_{ϕ_2}, $M_{\phi_1^{2n_2}}$ covers both M_{ϕ_1} and M_{ϕ_2}. It follows that M_{ϕ_1} and M_{ϕ_2} are commensurable. □

Lemma 2.7. *For a Haken manifold M, M is achiral if and only if -1 is a mapping degree of M.*

Proof. Suppose $f : M \to M$ is a map of degree -1. Then f induces a surjection on $f_* : \pi_1(M) \to \pi_1(M)$ [He1, page 175]. Since M is Haken, $\pi_1(M)$ is Hopfian, which first implies that f_* is an isomorphism [He1, page 177], [He2], then implies that f is homotopic to a homeomorphism [He1, Chap. 13], which must be orientation reversing. Another direction is obvious. □

Lemma 2.8. *Each torus semi-bundle N_ψ has a unique torus bundle double cover M_ϕ. The covering is characteristic and has a realization as $M_\phi \to N_\psi$ with $\psi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\phi = \begin{pmatrix} 2bc+1 & 2ab \\ 2cd & 2bc+1 \end{pmatrix}$.*

Proof. It is known that each N_ψ is doubly covered by a torus bundle M_ϕ ([Ha], Theorems 2.6 and 2.8, see also [SWW] Theorems 2.3 and 2.4). We just verify that M_ϕ is unique and characteristic.

For each torus semi-bundle N_ψ, we have a short fiber exact sequence

$$1 \to \pi_1(T) \to \pi_1(N_\psi) \to \mathbb{Z}_2 * \mathbb{Z}_2 \to 1. \qquad (2.4)$$

The unique torus bundle double cover $f : M_\phi \to N_\psi$ is given by the unique index 2 subgroup $\mathbb{Z} \subset \mathbb{Z}_2 * \mathbb{Z}_2$, which is also characteristic, indeed $\mathbb{Z}_2 * \mathbb{Z}_2$ has only three index 2 subgroups, two of them isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$, one is isomorphic to $\mathbb{Z}$. There is an induced commutative diagram between of these exact sequences.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(T) & \xrightarrow{i'_*} & \pi_1(M_\phi) & \xrightarrow{q'_*} & \mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle f|_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \bar{f}_*} & & \\ 1 & \longrightarrow & \pi_1(T) & \xrightarrow{i_*} & \pi_1(N_\psi) & \xrightarrow{q_*} & \mathbb{Z}_2 * \mathbb{Z}_2 & \longrightarrow & 1 \end{array} \qquad (2.5)$$

where $f|_*$ is an isomorphism. Then we can identify $\pi_1(T) \subset \pi_1(M_\phi)$ and $\pi_1(T) \subset \pi_1(N_\psi)$. Note any homeomorphism on τ on N_ψ keeps $T = \partial N$ invariant up to isotopy and $\pi_1(T)$ is a normal subgroup of $\pi_1(N_\psi)$. It follows that $\pi_1(T) \subset \pi_1(N_\psi)$ is characteristic. Then for any automorphism η on $\pi_1(N_\psi)$, η induces an automorphism $\bar{\eta}$ on $\mathbb{Z}_2 * \mathbb{Z}_2$. Since $\bar{\eta}(\mathbb{Z}) = \mathbb{Z}$, we have $q^{-1}(\bar{\eta}(\mathbb{Z})) =$

$q^{-1}(\mathbb{Z})$. Since $q^{-1}(\bar{\eta}(\mathbb{Z})) = \eta(q^{-1}(\mathbb{Z}))$, we have $\eta(q^{-1}(\mathbb{Z})) = q^{-1}(\mathbb{Z})$. That is $\pi_1(M_\phi) = q^{-1}(\mathbb{Z})$ is characteristic. □

Lemma 2.9. *$\mathcal{M}$ is achiral if and only if each element in $\mathcal{M}$ is virtually achiral.*

Proof. Suppose $\mathcal{M}$ is achiral, we may assume that M is achiral, that is there is an orientation reversing homeomorphism τ on M. For any $M' \in \mathcal{M}$, let $\tilde{M}$ be a common finite cover of M and M'. Suppose the covering degree $\tilde{M} \to M$ is d, then the index $[\pi_1(M) : \pi_1(\tilde{M})] = d$. Since $G = \pi_1(M)$ is finitely generated, by group theory G has only finitely many subgroup of indes d. Let

$$H = \bigcap_{\{G_i, [G:G_i]=d\}} G_i.$$

Then H is a finite index charecteristic subgroup of G. Let $M_H \to M$ be the covering corresponding to H. Since $\tau_*(H) = H$, τ lifts to a map τ_H on M_H, which must be an orientation reversing homeomorphism, that is M_H is achiral. Clearly M_H covers $\tilde{M}$, so covers M_2. That is M_2 is virtually achiral. We prove one direction.

Another direction is from the definition. □

3. Achirality, Gauss' Theory of Class Groups and Genus

Recall a connected closed oriented manifold is called achiral if it admits an orientation reversing homeomorphism.

Lemma 3.1. *Let $M = M_\phi$ be an oriented Sol torus bundle. Then M_ϕ is achiral if and only if there exists $A \in \mathrm{GL}_2(\mathbb{Z})$ such that one of the following holds.*

(i) $\phi A = A\phi$, $\det A = -1$.
(ii) $\phi A = A\phi^{-1}$, $\det A = 1$.

Proof. Let f be a self homeomorphism on M_ϕ. By Lemma 2.5, we assume that f is fiber preserving, so f is a fiber preserving horizontal covering, and (2.3) in the proof of Lemma 2.6 become

$$A \circ \phi = \phi^\epsilon \circ A,$$

where $A \in \mathrm{GL}(\mathbb{Z})$ since the restriction of f on thr fiber T is a homeomorphism.

It is clear that $\epsilon \cdot \det A = -1$ if and only if f is orientation reversing. □

We now recall Gauss' of class group theory [Gauss], [Fl, Chapter 5], [EW, Chapters 3, 4].

Recall $\mathrm{SL}_2(\mathbb{Z})$ acts on the complex plane by

$$\phi z = \frac{az+b}{cz+d}, \qquad \forall \phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),\ z \in \mathbb{C},$$

which induces an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{R}$. There are two conjugate real quadratic numbers $z_\phi, \bar{z}_\phi$ fixed by ϕ when $|\mathrm{tr}\phi| > 2$.

Let $M = M_\phi, \phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $|\mathrm{tr}\phi| > 2$. We define the discriminant of ϕ and of M to be

$$D_M = D_\phi := \frac{(d+a)^2 - 4}{u^2} > 0, \qquad (3.1)$$

It is easy to see that $D = D_\phi \equiv 0, 1 \bmod 4$ is a non-square positive integer. Let $D \equiv 0, 1 \bmod 4$ be a non-square integer. Define primitiive binary quadratic form Q_ϕ with discrimiment D given by

$$Q_\phi(x, y) := \mathrm{signtr}(\phi) \frac{cx^2 + (d-a)xy - by^2}{\gcd(c, d-a, b)} = \alpha x^2 + \beta xy + \gamma y^2, \qquad (3.2)$$

where $\alpha = \frac{c}{u}, \beta = \frac{d-a}{u}$, $\gamma = \frac{-b}{u}$, $u = \mathrm{sign}\,\mathrm{tr}(\phi)\gcd(b, c, (d-a))$. Note $\alpha x^2 + \beta x + \gamma$ is a minimal polynomial of the fixed point of ϕ, which is unique up to a sign, and

$$D_\phi = \beta^2 - 4\alpha\gamma > 0$$

is determined by the fixed point of ϕ. Since ϕ and $\phi^n, n \neq 0$ has the same fixed points, we have D_{ϕ^n} is invariant for all $n \neq 0$.

Lemma 3.2. *For each $\phi \in SL_2(\mathbb{Z})$, $Q_{-\phi} = Q_\phi$, $Q_{\phi^{-1}} = -Q_\phi$, $Q_{w\phi w}((x,y)w) = -Q_\phi(x,y)$.*

Proof. Suppose $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $-\phi = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$, $\phi^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and $w\phi w = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$. Then the lemma follows from a direct calculation. □

For each $A \in \mathrm{GL}_2(\mathbb{Z})$ one can verify that

$$Q_{A\phi A^{-1}}(x,y) = Q_\phi((x,y)(A^t)^{-1}). \qquad (3.3)$$

Call two primitive binary quadratic forms Q_1 and Q_2 with discriminant D are $SL_2(\mathbb{Z})$-equivalent, if there exists $A \in SL_2(\mathbb{Z})$, such that

$$Q_1((x,y)) = Q_2((x,y)A).$$

Those $SL_2(\mathbb{Z})$-equivalent classes of primitive binary quadratic forms of discriminant D is the class group $C(D)$. For a quadratic form Q, denote by $[Q]$ its class in $C(D)$. Let $Q_{0,D}$ denote the principal form of discriminant D, i.e.

$$x^2 - \frac{D}{4}y^2, \text{ (if } 4|D \text{);} \qquad x^2 + xy + \frac{1-D}{4}y^2 \text{ (if } 4 \nmid D \text{).}$$

Then $[Q_{0,D}]$ is the zero element in $C(D)$. For any form $Q = (\alpha, \beta, \gamma) := \alpha x^2 + \beta xy + \gamma y^2$, let $-Q = (-\alpha, -\beta, -\gamma)$, for $[Q] \in C(D)$, let $-[Q]$ denote its negative in $C(D)$.

Lemma 3.3. *For forms of discriminant D,*
(i) $[Q_1] = -[Q_2]$ if and only if $Q_1(x,y) = Q_2((x,y)B)$, where $B \in \mathrm{GL}_2(\mathbb{Z})$ and $\det B = -1$;
(ii) $[-Q] = -[Q] + [-Q_{0,D}]$;
(iii) $[-Q] = \pm[Q]$ is equivalent to either $[Q_{0,D}] = [-Q_{0,D}]$ or $2[Q] = [-Q_{0,D}]$;
(iv) $[Q] = [-Q_{0,D}]$ is equivalent to Q represents -1;
(v) $[-Q_{0,D}] = [Q_{0,D}]$ is equivalent to $x^2 - Dy^2 = -4$ has integral solution.

Theorem 3.4. *Let $M = M_\phi$ be an oriented Sol torus bundle with $D = D_\phi$ defined in (3.1) and Q_ϕ defined in (3.2). The following are equivalent.*
(1) *the Sol 3-manifold M is achiral.*
(2) *either $[Q_{0,D}] = [-Q_{0,D}]$ or $2[Q_\phi] = [-Q_{0,D}]$ (or quivalently $[-Q_\phi] = \pm[Q_\phi]$).*

Proof. Assume that (1) M_ϕ is achiral. With notations in Lemma 3.1, let

$$A = \begin{pmatrix} x & z \\ y & w \end{pmatrix}, \phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The conditions (i) and (ii) imply that

$$A = \begin{cases} \begin{pmatrix} x & \frac{by}{c} \\ y & \frac{(d-a)y+cx}{c} \end{pmatrix}, & \det A = -1, \qquad \text{for (i)} \\ \begin{pmatrix} x & \frac{(d-a)x-by}{c} \\ y & -x \end{pmatrix}, & \det A = 1 \qquad \text{for (ii).} \end{cases}$$

It is easy to see that the condition (i) is equivalent to that $Q_{0,D}$ represents -1, therefore equivalent to $[Q_{0,D}] = [-Q_{0,D}]$ by Lemma 3.3 (iv).

The condition (ii) is equivalent to that the following equation has integral solution

$$x^2 + yz = -1, \qquad \alpha z = \beta x + \gamma y, \qquad (3.4)$$

where $\alpha = \frac{c}{u}, \beta = \frac{d-a}{u}, \gamma = \frac{-b}{u}$, $u = \text{signtr}(\phi)\gcd(b, c, (d-a))$. It follows that

$$\begin{pmatrix} y & x \\ x & -z \end{pmatrix} \begin{pmatrix} \gamma & \beta/2 \\ \beta/2 & \alpha \end{pmatrix} \begin{pmatrix} y & x \\ x & -z \end{pmatrix} = \begin{pmatrix} -\alpha & \beta/2 \\ \beta/2 & -\gamma \end{pmatrix}, \qquad (3.5)$$

Recall the one-one correspondence of the form $\alpha x^2 + \beta xy + \gamma y^2$ and its matrix $\begin{pmatrix} \alpha & \beta/2 \\ \beta/2 & \gamma \end{pmatrix}$. Let

$$Q_1 = \gamma x^2 + \beta xy + \alpha y^2,\ Q_2 = -\alpha x^2 + \beta xy - \gamma y^2$$

By (3.5) we have

$$Q_2(x, y) = Q_1((x, y)B),$$

where $B = \begin{pmatrix} y & x \\ x & -z \end{pmatrix}$. By (3.4), $B \in SL_2(\mathbb{Z})$, so $[Q_1] = [Q_2]$.

Note $Q_\phi(x, y) = \alpha x^2 + \beta xy + \gamma y^2$, and we also have

$$Q_1(x, y) = Q_\phi((x, y)B_1),\ Q_2(x, y) = -Q_\phi((x, y)B_2),$$

where $B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Since the determinants of both B_1 and B_2 is -1, we have $[Q_1] = -[Q_\phi]$ and $[Q_2] = -[-Q_\phi]$ by Lemma 3.3 (i). Those two equalities together with $[Q_1] = [Q_2]$ implies that $[Q_\phi] = [-Q_\phi]$. By Lemma 3.3 (ii), we have $[Q_\phi] = -[Q_\phi] + [-Q_{0,D}]$, or equivalently $2[Q_\phi] = [-Q_{0,D}]$. □

We now recall Gauss' genus theory (see [Gauss], [Fl] and [BS]). Let $D \equiv 0, 1 \bmod 4$ be a non-square integer, the Kronecher symbol $\chi_D : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$ is a homomorphism such that $\chi_D(p \bmod D) = \left(\frac{D}{p}\right)$ for all primes $p \nmid 2D$, where $\left(\frac{D}{p}\right)$, the Legendre symbol, is 1 if and only if D is quadratic residue modulo p. Let $H_D \subset \text{Ker}\chi_D$ be the subgroup consisting of values represented by the principal form of discriminant D. Then there is an exact sequence

$$0 \to 2C(D) \longrightarrow C(D) \xrightarrow{\omega} \text{Ker}\chi_D/H_D \longrightarrow 0, \qquad (3.6)$$

where ω sends a class to the coset of H_D in $\text{Ker}\chi_D$ it represents.

The group H_D consists of elements $[a] \in (\mathbb{Z}/D\mathbb{Z})^\times$ such that

$$\left(\frac{a}{p}\right) = 1 \text{ for all odd prime divisors } p \text{ of } D \qquad (3.7)$$

and such that

$$a \equiv \begin{cases} 1, 7 \bmod 8, & \text{if } D \equiv 8 \bmod 32, \\ 1 \bmod 4, & \text{if } D \equiv 12, 16, 28 \bmod 32, \\ 1, 3 \bmod 8, & \text{if } D \equiv 24 \bmod 32, \\ 1 \bmod 8, & \text{if } 32|D. \end{cases} \qquad (3.8)$$

Theorem 3.5. *Let $D \equiv 0, 1 \bmod 4$ be a positive non-square integer. Then the following are equivalent:*

(1) *there exists an achiral oriented Sol torus bundle M of discriminant D,*
(2) *$16 \nmid D$ and no prime factor of D is $\equiv 3 \bmod 4$.*

Proof. We first verity that (1) is equivalent to the following

(3) there exists a $[Q] \in C(D)$ such that $2[Q] = [-Q_{0,D}]$ (or equivalently $2[Q]$ represents -1).

The equivalence of (1) and (3) follows from Theorem 3.4: If there exists a $[Q] \in C(D)$ such that $2[Q] = [-Q_{0,D}]$, then the Sol 3-manifold M_ϕ with $Q_\phi = Q$ is achiral by Theorem 3.4. On the other hand if M_ϕ with $D_\phi = D$ is achiral, then either $[Q_{0,D}] = [-Q_{0,D}]$ or $2[Q_\phi] = [-Q_{0,D}]$ by Theorem 3.4. Note if $[Q_{0,D}] = [-Q_{0,D}]$, choose ϕ_0 such that $Q_{\phi_0} = Q_{0,D}$. Then $2[Q_{\phi_0}] = 2[Q_{0,D}] = [Q_{0,D}] = [-Q_{0,D}]$.

Below we will prove that (2) and (3) are equivalent in two steps by using Gauss' genus theory.

Step 1: To verify that (3) $2[Q] = [-Q_{0,D}]$ ($2[Q]$ represents -1) is equivalent to $[-1] \in H_D$ by using the short exact sequence (3.6):

Suppose -1 is represented by $2[Q]$. Since $2[Q] \in 2C(D)$, we have $\omega(2[Q]) = \bar{1} \in \mathrm{Ker}\chi_D/H_D$, which implies $[-1] \in H_D$.

Suppose $[-1] \in H_D$. We only need to prove $\omega([-Q_{0,D}]) == \bar{1} \in \mathrm{Ker}\chi_D/H_D$, therefore $[-Q_{0,D}] = 2[Q]$ for some $[Q] \in C(D)$. If not, we have $\omega([-Q_{0,D}]) = gH_D \in \mathrm{Ker}\chi_D/H_D$, where $g \notin H_D$. Since $[-1] \in H_D$, $[-1] \notin gH_D$, which implies $-Q_{0,D}$ does not represent any $a \equiv -1 \bmod D$, contradicting that $-Q_{0,D}$ represents -1.

Step 2: To verify $[-1] \in H$ and (2) are equivalent by using (3.7) and (3.8).

For any odd prime $p|D$, by Euler's Theorem, $(\frac{-1}{p}) = 1$ if and only if p is not $\equiv 3 \bmod 4$.

Suppose $[-1] \in H_D$. By Euler's Theorem and (3.6) (a charaterization of H_D), we have that no prime factor of D is $\equiv 3 \bmod 4$. Since -1 is not $\equiv 1 \bmod 8$, by the fourth line of (3.7) (another charaterization of H_D), we have D is not a multiple of 32. Since -1 is not $\equiv 1 \bmod 4$, by the second line of (3.7), we have D is not congruent to 16 mod 32. Hence D is not a multiple of 16.

Now suppose $16 \nmid D$ and no prime factor of D is $\equiv 3 \bmod 4$. Then no prime factor of D is $\equiv 3 \bmod 4$ implies that for any odd prime $p|D$, $\left(\frac{-1}{p}\right) = 1$ [Fl, Page 71]. Moreover suppose

$$D = 2^\alpha \prod_{i=1}^{k} (4n_i + 1)^{\beta_i}$$

is the prime decomposition of D. Then $D = 2^\alpha(4n+1)$ for some $n \in \mathbb{Z}$. Since $16 \nmid D$, we have $\alpha \leq 3$. We have $4 \nmid D$ if $\alpha \leq 1$, and $D \equiv 4 \bmod 16$ if $\alpha = 2$, and $D \equiv 8 \bmod 32$ if $\alpha = 3$. Overall we have that D is not $0, 16, 12, 24, 28,$ mod 32. Then by -1 appears in (3.7) for other D. And by the charaterization of H_D, $[-1] \in H_D$. □

Corollary 3.6. *A commensurable class $\mathcal{M}$ of Sol 3-manifolds is achiral if and only $D_\mathcal{M}$ contains no prime $\equiv 3 \bmod 4$.*

Proof. Suppose $\mathcal{M}$ is achiral, then there is achiral Sol 3-manifold $M' \in \mathcal{M}$. Then $D_{M'}$ contains no prime $\equiv 3 \bmod 4$ by Theorem 3.5. Since $D_{M'} = c^2 D_\mathcal{M}$ ($c \in \mathbb{Z}$), $D_\mathcal{M}$ contains no prime $\equiv 3 \bmod 4$.

On the other hand, from (2.1), the definition of $D_\mathcal{M}$, we know that $16 \nmid D_\mathcal{M}$. If $D_\mathcal{M}$ contains no prime $\equiv 3 \bmod 4$, then there is an achiral Sol 3-manifold $M' \in \mathcal{M}$ with $D_{M'} = D_\mathcal{M}$ by Theorem 3.5. So $\mathcal{M}$ is achiral. □

Corollary 3.7. *Among Sol 3-manifolds:*

(1) Each commensurable class contains non-achiral manifolds.

(2) There are infinitely many achiral commensurable classes, however their density among all commensurable class is zero ordered by discriminants.

Proof. (1) For each commensurable class $\mathcal{M}$ of Sol 3-manifolds and each integer c, there is a $M' \in \mathcal{M}$ such that $D_{M'} = c^2 D_\mathcal{M}$. If we choose c to be either 3 or 4, then M' is non-achiral by Theorem 3.5.

(2) Let D be prime in the form of $4k+1$. The $D = D_\mathcal{M}$ for some commensurable class $\mathcal{M}$ by Proposition 2.4, and $\mathcal{M}$ is achiral by Corollary 3.6. It is a well known fact that there are infinitely many primes in this form of $4k+1$ [EW, Theorem 10.5].

Now we prove the "however" part. Note that a integer $n > 0$ is the sum of two integral squares if and only if the square-free part of n contains no prime $\equiv 3 \bmod 4$ [Fl, Chap. 2]. So we have

$$\{D_{\mathcal{M}} | D_{\mathcal{M}} \text{ contains no prime} \equiv 3 \bmod 4\} = \{D_{\mathcal{M}} | D_{\mathcal{M}} = a^2 + b^2\}.$$

Furthermore

$$\lim_{X\to\infty} \frac{\#\{0 < D < X | D = a^2 + b^2\}}{X} = 0,$$

$$\lim_{X\to\infty} \frac{\#\{D_{\mathcal{M}} < X\}}{X} = \lim_{X\to\infty} \frac{\#\{D < X | D \equiv 0, 1 \bmod 4, D \text{ fundament}\}}{X} = C > 0$$

(indeed $C = 3/\pi^2$), so we have

$$\lim_{X\to\infty} \frac{\{D_{\mathcal{M}} | D_{\mathcal{M}} \text{ contains no prime} \equiv 3 \bmod 4\}}{\#\{D_{\mathcal{M}} < X\}} = 0$$

and the "however" part follows from Corollary 3.6. □

4. A Density Result

Proposition 4.1. *Let $D \equiv 0, 1 \bmod 4$ be a positive non-sqaure integer. Then the following are equivalent:*

(1) *The negative Pell equation $x^2 - Dy^2 = -4$ has solutions;*
(2) *There is a non-oriented Sol 3-manifold with characteristic double cover of discriminant D;*
(3) *There is an achiral semi-tours bundle with characteristic double cover of discriminant $\operatorname{lcm}(D, 4)$.*

Proof. Assume (1), or equivalently, by Lemma 3.3 (iv) and (v), the solvability of

$$u^2 - \frac{D}{4}v^2 = -1 \quad (4|D), \qquad u^2 - Dv^2 = -1 \quad (4 \nmid D). \qquad (4.1)$$

Here for the later one, note that if $x^2 - Dy^2 = -4$ has solution then has solution with x even. Let

$$\psi = \begin{pmatrix} u & \frac{D}{4}v \\ v & u \end{pmatrix} \quad (4|D), \qquad \begin{pmatrix} u - v & \frac{D-1}{2}v \\ 2v & u + v \end{pmatrix} \quad (4 \nmid D). \qquad (4.2)$$

Then the determinent of ψ is -1 by (4.1), so the torus bundle M_ψ is non-orientable. It is the direct calculation that $D_\psi = D$. Then $M = M_{\psi^2}$ is the unique orientable double cover of M_ψ. Since ψ^2 and ψ have the same fixed points as actions on $\mathbb{R} \cup \infty$, we have $D_\psi^2 = D_\psi$ by definition. So $D_M = D$.

On the other hand, assume (2), i.e. the double cover M_{ψ^2} of M_ψ, $\det \psi = -1$, has discriminant D, then by the equality

$$\psi^2 = (w\psi)^{-1} w\psi^2 w (w\psi), \qquad (4.3)$$

where $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with $\det w = -1$. Since $\det w\psi = 1$, by (4.3) we have

$$[Q_{\psi^2}] = [Q_{w\psi^2 w}]. \qquad (4.4)$$

By Lemma 3.2, we have

$$-Q_{\psi^2}(x, y) = Q_{w\psi^2 w}(y, x)$$

which implies that

$$[-Q_{\psi^2}] = -[Q_{w\psi^2 w}] \qquad (4.5)$$

Combing (4.4), (4.5) and $[-Q] = -[Q] + [-Q_{0,D}]$ for and $[Q] \in C(D)$ (Lemma 3.3) (iii)), we have $[-Q_{0,D}] = 0$, that is $[-Q_{0,D}] = [Q_{0,D}]$, which is equivalent to the negative Pell equation

$x^2 - Dy^2 = -4$ has integral solution (Lemma 3.3) (ii)). This shows the equivalence between (1) and (2).

Note that the double cover M_ϕ of N_ψ is given as follows:

$$\phi = \begin{pmatrix} 2bc+1 & 2bd \\ 2ac & 2bc+1 \end{pmatrix}, \qquad \text{if } \psi = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ abcd \neq 0.$$

It is clear that M_ϕ has discriminant $D_\phi = 4\frac{abcd}{\gcd(ac,bd)^2}$. Note that

- $2[Q_\phi] \in C(D_\phi)$ is the zero class by Proposition 3.5, and thus M_ϕ is achiral if and only if the negative equation $x^2 - D_\phi y^2 = -4$ has solution.
- N_ψ is achiral if and only -1 is a self-mapping degree of N_ψ by Lemma 2.7, and if and only if $a+d=0$ by [SWW, Theorem 1.7].
- if $a+d=0$, then $D_\phi = -\frac{4bc}{\gcd(b,c)^2}$, and therefore $-4 = 4a^2+4bc = 4a^2 - D_\phi \gcd(b,c)^2$.
- if if N_ψ is achiral, then M_ϕ is achiral, since the cover $M_\phi \to N_\psi$ is characteristic by the "Moreover" part of Proposition 2.1

It follows that if N_ψ is achiral with $D_\phi = \mathrm{lcm}(D,4)$, then $x^2 - Dy^2 = -4$ has solutions.

On the other hand, if the negative Pell equation $x^2 - Dy^2 = -4$ has solution, then there is always solution with x even, say $(2a,b)$. Take

$$\psi = \begin{cases} \begin{pmatrix} a & b \\ -Db/4 & -a \end{pmatrix}, & \text{if } 4|D, \\ \begin{pmatrix} a & b/2 \\ -Db/2 & -a \end{pmatrix}, & \text{if } D \equiv 1 \bmod 4. \end{cases}$$

Then M_ϕ is achiral of discriminant D or $4D$. This shows the equivalence between (1) and (3). ☐

Corollary 4.2. *Let $\mathcal{M}$ be a commensurable class of Sol 3 manifolds of discriminant D. Then the following conditions are equivalent.*

1. *the negative Pell equation $x^2 - Dy^2 = -4$ has solutions;*
2. *$\mathcal{M}$ contains a non-orientable Sol 3-manifolds;*
3. *$\mathcal{M}$ contains an achiral tour semi-bundle.*

The equation $x^2 - Dy^2 = -4$ is called the negative Pell equation. The solvability of negative Pell equation has a long history and P Stevenhagen [Ste] made the following conjecture.

Conjecture 4.3 (Stevenhagen). *The density of fundamental positive discriminants D, for which $x^2 - Dy^2 = -4$ have solutions, among all fundamental positive discriminants D without prime factors $\equiv 3 \bmod 4$ is $1-\rho$ with*

$$\rho := \prod_{j=1}^{\infty} \left(1+2^{-j}\right)^{-1} = 0.41942\cdots.$$

This conjecture is recently proved by P. Koymans and C. Pagano [KP]. It is a big surprising that the Stevenhagen's conjecture has its avatar in topology, which we will state now, and which follows from Corollary 3.6, Corllary 4.2 and [KP].

Theorem 4.4. *Among all achiral commensurable classes of Sol 3-manifolds ordered by discriminants, the density of classes containing non-orientable elements:*

$$\lim_{X\to\infty} \frac{\#\{D < X \mid \mathcal{M}_D \textit{ contains an non-orientable element}\}}{\#\{D < X \mid \mathcal{M}_D \textit{ is achiral}\}} = 1-\rho,$$

where

$$\rho := \prod_{j=1}^{\infty} \left(1+2^{-j}\right)^{-1} = 0.41942\cdots.$$

Proposition 4.5. *There exists $M \in \mathrm{Sol}_D$ double cover a tours semi-bundle if and only if $4|D$. Moreover, $M_\phi \in \mathrm{Sol}_D$ is the double cover of a tours semi-bundle if and only if $2[Q_\phi] = 0$ and $\phi \equiv I \bmod 2\mathbb{Z}$, in this case, there is a unique tours semi-bundle double covered by M_ϕ.*

Proof. By [Ma], $M_\phi \in \mathrm{Sol}_D$ is the double of a tours semi-bundle if and only if there exists $A \in \mathrm{GL}_2(\mathbb{Z})$ such that

$$A\phi = \phi^{-1}A, \qquad \det A = -1, \qquad A \equiv \phi \equiv I \bmod 2\mathbb{Z}.$$

Let $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ and assume that $Q_\phi = \alpha x^2 + \beta xy + \gamma y^2$. Then the condition $A\phi = \phi^{-1}A$ and $\det A = -1$ is equivalent to

$$A = \begin{pmatrix} x & y \\ z & -x \end{pmatrix}, \quad x^2 + yz = 1, \quad \alpha z = \beta x + \gamma y.$$

One can check that

$$\begin{pmatrix} -x & y \\ -z & -x \end{pmatrix} \begin{pmatrix} \alpha & -\beta/2 \\ -\beta/2 & \gamma \end{pmatrix} \begin{pmatrix} -x & -z \\ y & -x \end{pmatrix} = \begin{pmatrix} \alpha & \beta/2 \\ \beta/2 & \gamma \end{pmatrix}.$$

It follows that the condition that there exists $A \in \mathrm{GL}_2(\mathbb{Z})$ such that $A\phi = \phi^{-1}A$ and $\det A = -1$ if and only if $2[Q_\phi] = 0$.

Note that $\phi \equiv I \bmod 2$ implies $4|D$. Assume that $4|D$ take $M_\phi \in \mathrm{Sol}_D$ such that $2[Q_\phi] = 0$, we have that ϕ is $\mathrm{SL}_2(\mathbb{Z})$-conjugate to $\phi_0 = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$. It is easy to see that $A_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv I \bmod 2$ satisfies $A_0\phi_0 = \phi_0^{-1}A_0$, $\det A_0 = -1$, and that $\phi_0^2 \equiv I \bmod 2$. Thus $M_{\phi^2} \in \mathrm{Sol}_D$ is the double cover of a tours semi-bundle.

Let A, A' be two such matrices, then $A' = AC$ with $C \in \mathrm{SL}_2(\mathbb{Z}), \equiv I \bmod 2$ and $C\phi = \phi C$. One can easily check that A' is equivalent to A in the sense of [Ma]. Thus if M_ϕ is the double cover of at most one tours semi-bundle. ☐

5. Shimizu L-functions

In the 1970s, Hirzebruch [Hr] formulated a conjecture on the signature of the Hilbert modular varieties whose cusp cross-sections are solvmanifolds, which relates the signature defects of these Sol manifolds to special values of Shimizu's L-functions of totally real fields. We now recall the definition of Shimizu L-functions for Hilbert modular surfaces. In this case, these solvmanifolds are oriented tours bundle M_ϕ.

Given $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with discrimimant $D_\phi = D$ and quadratic form Q_ϕ:

the Shimizu L-function is defined as

$$L(M,s) := \sum_{(0,0)\neq(x,y)\in\mathbb{Z}^2/\phi^{\mathbb{Z}}} \frac{\mathrm{sign}\, Q_\phi(x,y)}{|Q_\phi(x,y)|^s}, \qquad \mathrm{Re}(s) > 1$$

only depends on its oriented equivalence class. In fact $L(M,s)$ has an entire continuation to $\mathbb{C}$ (See Proposition 5.6).

Theorem 5.1. *Let $M = M_\phi$ be an oriented Sol torus bundle.*

(1) M is achiral if and only if the Shimizu L-function $L(M,s) \equiv 0$.

(2) If $L(M_{\phi_1}, s) = L(M_{\phi_2}, s)$ for two non-achiral oriented bundles M_{ϕ_1}, M_{ϕ_2} with same discriminant, then M_{ϕ_1} is homeomorphic to either M_{ϕ_2} or $M_{-\phi_2}$ as oriented torus bundle.

We list some facts for the proof of Theorem 5.1.

Proposition 5.2. [Web] *Each $[Q] \in C(D)$ represents infinitely many primes.*

Lemma 5.3. *It both $Q_1, Q_2 \in C(D)$ represent a prime p with $(p, 2D) = 1$, then $[Q_1] = \pm[Q_2]$.*

Proof. By [EW, Exercise 3.30], each Q_i is $SL_2(\mathbb{Z})$-equivalent to a form $Q_i'(x,y) = px^2 + d_i xy + e_i y^2$, $d_i \le p$. Note $d_i - 4pe_i = D$. Clearly d_1 and d_2 has the same parity. Therefore $\frac{|d_1|-|d_2|}{2}, \frac{|d_1|+|d_2|}{2} \in \mathbb{Z}$. By $p \nmid D$, we have $p \nmid d_i$ and therefore $0 < |d_i| < p$. Therefore $0 < \frac{|d_1|+|d_2|}{2} < p$. Since

$$\frac{d_1^2 - d_2^2}{4} = \frac{|d_1| - |d_2|}{2} \frac{|d_1| + |d_2|}{2} = p(e_1 - e_2).$$

Therefore $p | \frac{|d_1|-|d_2|}{2}$. It follows $|d_1| = |d_2|, e_1 = e_2$. So $[Q_1'] = \pm[Q_2']$ and $[Q_1] = \pm[Q_2]$. □

Lemma 5.4. *Suppose $\phi, \psi \in \mathrm{SL}_2(\mathbb{Z})$ with $D_\phi = D_\psi = D$. If $[Q_\phi] = \pm[Q_\psi]$ and $\mathrm{tr}(\phi) = \mathrm{tr}(\psi)$. Then ϕ is $\mathrm{SL}_2(\mathbb{Z})$-conjugate to either ψ or $w\psi^{-1}w$.*

Proof. (i) First note that ϕ is determined by Q_ϕ and $\mathrm{tr}(\phi)$: Suppose $Q_\phi(x,y) = \alpha x^2 + \beta xy + \gamma y^2$, one can solve that $\phi = \mathrm{sign}\mathrm{tr}(\phi) \begin{pmatrix} \frac{t+\beta s}{2} & -\gamma s \\ \alpha s & \frac{t-\beta s}{2} \end{pmatrix}$, where $t = |\mathrm{tr}(\phi)|$ and $s = \sqrt{\frac{t^2-4}{D}}$.

(ii) If $[Q_\phi] = [Q_\psi]$, then $Q_\phi(x,y) = Q_\psi((x,y)A)$; If $[Q_\phi] = -[Q_\psi]$, then $Q_\phi(x,y) = Q_\psi((x,y)A'w)$; both $A, A' \in \mathrm{SL}_2(\mathbb{Z})$. Then one can verifies the Lemma by (i), (3.3) and Lemma 3.2. □

By Lemma 3.2 we have

Lemma 5.5. *$L(M_{-\phi}, s) = L(M_\phi, s)$, $L(M_{\phi^{-1}}, s) = -L(M_\phi, s)$ and $L(M_\phi, s) = -L(M_{w\phi w}, s)$.*

Proof of Theorem 5.1. Suppose M is achiral, then $M_\phi = -M_\phi$, and furthermore $M_\phi = M_{\phi^{-1}}$ by Lemma 2.2. Then ϕ^{-1} is $SL_2(\mathbb{Z})$ conjugate to either ϕ or $w\phi^{-1}w$ by Lemma 2.3. That is either

$$L(M_{\phi^{-1}}, s) = L(M_\phi, s) \quad \text{or} \quad L(M_{\phi^{-1}}, s) = L(M_{w\phi^{-1}w}, s) \qquad (5.1)$$

Combining (5.1) and Lemma 5.5, we have $L(M, s) \equiv 0$.

On the other hand, let $K_{n,\phi} = \{(x,y) \in \mathbb{Z} \oplus \mathbb{Z} | Q(x,y) = n\}/\phi$ for each $n \in \mathbb{Z}$. $K_{n,\phi}$ is finite. In $L(M_\phi, s)$, the term $1/n^s$ occurs $|K_{n,\phi}|$ times since each orbit in $K_{n,\phi}$ contribute a single $1/n^s$. Similarly the term $-1/n^s$ occurs $|K_{-n,\phi}|$ times. So we have

$$L(M_\phi, s) = \sum_{n=1}^{\infty} \frac{-|K_{-n,\phi}| + |K_{n,\phi}|}{|n|^s}. \qquad (5.2)$$

Hence $L(M_\phi, s) \equiv 0$ if and only if $|K_{-n,\phi}| = |K_{n,\phi}|$ for each n, which is equivalent to Q and $-Q$ represent the same set of integers. Then both Q and $-Q$ represent some prime p with $(p, 2D) = 1$ by Proposition 5.2, and then $[-Q_\phi] = \pm[Q_\phi]$ by Lemma 5.3. By Theorem 3.4, this is equivalent to that M is achiral.

We have proved (1). Below we prove (2).

Suppose both $\mathrm{tr}(\phi_1), \mathrm{tr}(\phi_2) > 0$ (otherwise replace ϕ by $-\phi$). Let $\psi_1 = \phi_1^{n_1}$ and $\psi_2 = \phi_2^{n_2}$ such that $\mathrm{tr}\psi_1 = \mathrm{tr}\psi_2$, where n_i a positive integer, $i = 1, 2$. Then from the definition, it is easy to see

$$L(M_{\psi_i}, s) = n_i L(M_{\phi_i}, s), i = 1, 2. \qquad (5.3)$$

Following (5.2) and $L(M_{\phi_1}, s) = L(M_{\phi_2}, s)$ we have

$$L(M_{\psi_1}, s) = \frac{n_2}{n_1} L(M_{\psi_2}, s) = \sum_{n=1}^{\infty} \frac{-|K_{-n}| + |K_n|}{|n|^s} \qquad (5.4)$$

By Proposition 5.2, Q_{ψ_1} represents a prime p with $(p, 2D) = 1$. Then $-p$ is not represented by Q_{ψ_1} otherwise $[Q_{\psi_1}] = \pm[-Q_{\psi_1}]$ by Lemma 5.3, and therefore M_{ψ_1} is achiral by Theorem 3.4. So we have $K_p \neq 0$ and $K_{-p} = 0$. By (5.4) Q_{ψ_2} also represents p. Then $[Q_{\psi_2}] = \pm[Q_{\psi_1}]$ again by Lemma 5.3. Since $\mathrm{tr}\psi_1 = \mathrm{tr}\psi_2$, ψ_1 is $SL_2(\mathbb{Z})$-conjugated to either ψ_2 or $w\psi_2^{-1}w$ by Lemma 5.4. Therefore $L(\psi_1, s) = L(\psi_2, s)$ by Lemma 5.5. That is to say $n_1 = n_2$, and therefore ϕ_1 is $SL_2(\mathbb{Z})$-conjugated to either ϕ_2 or $w\phi_2^{-1}w$. Since M_{ϕ_2} and $M_{w\phi_2^{-1}w}$ are homeomorphic as oriented torus bundles, in either case, M_{ϕ_1} and M_{ϕ_2} are homeomorphic as oriented torus bundles.

When requiring $tr(\phi_i) > 0$, we replace ϕ_i by $-\phi_i$, so in general we have M_{ϕ_1} is homeomorphic to either M_{ϕ_2} or $M_{-\phi_2}$ as oriented torus bundles. □

Proposition 5.6. *For an oriented tours bundle $M = M_\phi$ of discriminant D, $L(M, s)$ has an entire continuation to $\mathbb{C}$ and satisfies a functional equation:*

$$\Lambda(M, s) := \Gamma_{\mathbb{R}}(s+1)^2 D^{s/2} L(M, s) = \Lambda(M, 1-s).$$

Proof. It follows (see [ADS]) that $L(M, s)$ ha s entire continuation and satisfies the functional equation:

$$\Lambda(M, s) = -\Lambda(M_{\phi^t}, 1-s).$$

Since for any $\phi \in \mathrm{SL}_2(\mathbb{Z})$, ϕ^t is $\mathrm{SL}_2(\mathbb{Z})$ conjugate to ϕ^{-1}:

$$\phi^t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

By Lemma 5.5, we have $L(M_{\phi^t}, s) = L(M_{\phi^{-1}}, s) = -L(M, s)$. So $\Lambda(M, s) = \Lambda(M, 1-s)$. □

References

[ADS] M.F.Atiyah, H.Donnelly, I.M.Singer, *Eta invariants, signature defects of cusps, and values of L-functions*, Annals of Math. 118 (1983), 131-177.

[BGV] A. P. Bareto, D. L. Goncalves and D. Vedruscolo, *Free involutions on torus semi-bundles and the Borsuk-Ulam theorem for maps into* $\mathbb{R}^n$, Hiroshima Math. J. **46** (2016), 255-270.

[BS] W. Bosma and P. Stevenhagen, *On the computation of quadratic 2-class groups*, Journal de Theorie des Nombres de Bordeaux, tome 8, n^o2 (1996), p.283-313.

[EW] G. Everest, T. Ward, *An introduction to number theory.* Graduate Texts in Mathematics, 232. Springer-Verlag London, Ltd., London, 2005.

[Ha] A. Hatcher, *Notes on basic* 3*-manifold topology.* http://pi.math.cornell.edu/~hatcher/.

[Fl] D. Flath, *Introduction to Number Theory*, Wiley, New York, 1988.

[Gauss] C.F. Gauss, *Disquisitiones Arithmeticae*, Gerhard Fleischer, Leipzig, 1801.

[He1] J. Hempel, 3*-manifolds,* Princeton University Press and University of Tokyo Press, 1976.

[He2] J. Hempel, *Residual finiteness for 3-manifolds. Combinatorial group theory and topology (Alta, Utah, 1984), 379-396, Ann. of Math. Stud., 111, Princeton Univ. Press, Princeton, NJ, 1987.*

[Hr] F. Hirzebruch, *Hilbert modular surfaces.* L'Enseign. Math. 19 (1973), 183-281.

[KP] P. Koymans and C. Pagano, *On Stevenhagen's conjecture*, arXiv: 2201.13424v1 [math.NT] 31 Jan 2022.

[LS] Y. Liu and H. Sun, *Toward and after virtual specialization in 3-manifold topology.* Surveys in differential geometry 2020. Surveys in 3-manifold topology and geometry, 215-252, Surv. Differ. Geom., 25, Int. Press, Boston, MA.

[Ma] B. Martelli *An Introduction to Geometric Topology* arXiv:1610.02592 math.GT math.DG (2016)

[Sak] M. Sakuma, *Involutions on torus bundles over* S^1, Osaka J. Math. **22** (1985), 163-185.

[Sc] P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1983), 401–487.

[Sh] H. Shimizu, *On discontinuous groups operating on the product of the upper half planes.* Ann. of Math. (2) 77 (1963), 33-71.

[Smi2] A. Smith. 2^∞*-Selmer groups,* 2^∞*-class groups, and Goldfeld's conjecture.* Preprint.

[Smi3] A. Smith, ℓ^∞*-Selmer groups in degree* ℓ *twist families.* Ph. D. Thesis.

[Ste] P. Stevenhagen, *The number of real quadratic fields having units of negative norm.* Experiment. Math. 2:121-136, 1993.

[SWW] H. Sun, S. Wang, J. Wu, *Self-mapping degrees of trus bundles and torus semi-bundles*, Osaka J. Math. **47** (2010), 131–155.

[Th] W. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Am. Math. Soc. 6 (1982), 357–381.

[Th1] W. P. Thurston, *Three-dimensional geometry and topology.* Vol. 1. Edited by Silvio Levy. Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997.
[Web] H. Weber, *Beweis des Satzes.* Math. Ann. 20 (1882), 301-329.
[Zhang] W.P. Zhang, *Lectures on Chern-Weil Theory and Witten Deformations*, Nankai Tracts in Mathematics Vol. 4, World Scientific.

Academy of Mathematics and System Science, Morningside Center of Mathematics, Chinese Academy of Sciences, Beijing 100190
Email address: ytian@math.ac.cn

Department of Mathematical Sciences, Peking University, Beijing 100871, CHINA
Email address: wangsc@math.pku.edu.cn

Department of Mathematical Sciences, Peking University, Beijing 100871 CHINA
Email address: wangzz22@stu.pku.edu.cn