Towards Three Problems of Katz on Kloosterman Sums

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Works to be presented

- P. Xi, When Kloosterman sums meet Hecke eigenvalues, *Invent. math.* **220** (2020), 61–127.
- P. Xi, Sign changes of Kloosterman sums with almost prime moduli. II, *Int. Math. Res. Not.* **4** (2018), 1200–1227.
- P. Xi, Gaussian distribution of Kloosterman sums: vertical and horizontal, *Ramanujan J.* **43** (2017), 493–511.
- P. Xi, Sign changes of Kloosterman sums with almost prime moduli, *Monatsh. Math.* **177** (2015), 141–163.

Kloosterman sums

$$\mathrm{Kl}(m,c)=\frac{1}{\sqrt{c}}\sum_{a\,(\mathrm{mod}\,c)}^{*}\mathbf{e}\Big(\frac{ma+\bar{a}}{c}\Big),$$

where $a\bar{a}\equiv 1\ (\mathrm{mod}\,c)$ and $\mathbf{e}(z)=\mathbf{e}^{2\pi iz}$.

- (1911, H. Poincaré):
  Fourier coefficients of modular functions (Poincaré series)
- (1926, H. D. Kloosterman):

$$N=ax^2+by^2+cz^2+dt^2$$

Weil's bound

Theorem (Weil, PNAS [1948] & Estermann, Mathematika [1961])

For each prime p, we have

$$|\,\mathrm{Kl}(m,p)\,| \leqslant 2.$$

In general,

$$|\,\mathrm{Kl}(m,c)\,| \leqslant \tau(c).$$

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Applications

- Divisor functions in APs (Selberg–Hooley), "$2+3$" (Selberg)
- Greatest prime factor of n^2+1 (Hooley)
- Brun–Titchmarsh theorem (Hooley, Iwaniec)
- Modular forms – Kloostermania (Kuznetsov, Deshouillers–Iwaniec, *et al*)
- Bounded gaps between primes (Zhang)

Katz's problems

Denote by $\mathcal{P}$ the set of primes. In [*Sommes Exponentielles*, Astérisque, Vol. 79 (1980)], Nicholas Katz proposed the following three problems (with $a \neq 0$ fixed):

(I) (sign change) *Does the density of* $\{p \in \mathcal{P} : \mathrm{Kl}(a,p) > 0\}$ *in* $\mathcal{P}$ *exist? If this is the case, is it equal to* $\frac{1}{2}$*?*

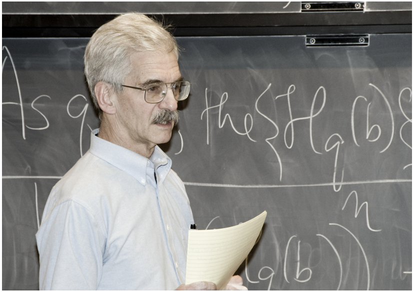
(II) (equidistribution) *Is there a measure on* $[-2,2]$ *such that* $\{\mathrm{Kl}(a,p) : p \in \mathcal{P}\}$ *equidistributes?*

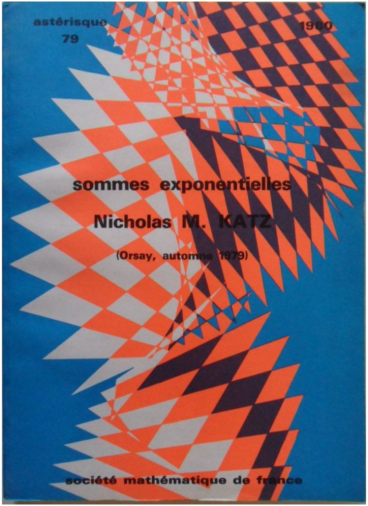
(III) (modular structure) *Consider the Euler product*

$$L_a(s) := \prod_{p\in\mathcal{P},\ p\nmid a} \left(1 - \frac{\mathrm{Kl}(a,p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1}$$

for $\Re s > 1$*. Is it defined to be an L-function attached to some Maass form of level* $\mathfrak{q}$ *with* $\mathfrak{q}$ *being a power of* 2*?*

Katz and his book





Katz's horizontal Sato–Tate conjecture (1980)

Clearly, (II) implies (I). Moreover, Katz formulated a precise conjecture.

Conjecture (Katz, Horizontal Sato-Tate Conjecture)

The set $\{\mathrm{Kl}(a,p) : p \in \mathcal{P}, p \nmid a\}$ *equidistributes in* $[-2,2]$ *w.r.t. the Sato–Tate measure* $\frac{1}{2\pi}\sqrt{4-x^2}\mathrm{d}x$ *for any fixed non-zero integer* a.

In other words, one has, for any interval $I \subset [-2,2]$ and $a \neq 0$,

$$\lim_{X\to+\infty} \frac{1}{\pi(X)}\#\{p \leqslant X : \mathrm{Kl}(a,p) \in I\} = \frac{1}{2\pi}\int_I \sqrt{4-x^2}\mathrm{d}x.$$

[Motivated by the Sato-Tate conjecture for elliptic curves, but much more difficult due to the constructions of related L-functions.]

Elliptic curves

Let $E/\mathbf{Q}$ be an elliptic curve given by $y^2 = x^3 + ax + b$ with $a, b \in \mathbf{Z}$. Denote by N the conductor of E. For each $p \in \mathcal{P}$, consider

$$N_p(E) := \#\{(x,y) \pmod p : y^2 \equiv x^3 + ax + b \pmod p\}$$

and define $a_p(E) = p + 1 - N_p(E)$. Hasse (1933) proved that $|a_p(E)| \leqslant 2\sqrt{p}$. The Hasse-Weil zeta function for E is *related* to the the Euler product

$$\prod_{p \nmid \text{conductor}} \Big(1 - \frac{a_p(E)}{p^s} + \frac{1}{p^{2s-1}}\Big)^{-1}, \quad \Re s > 3/2.$$

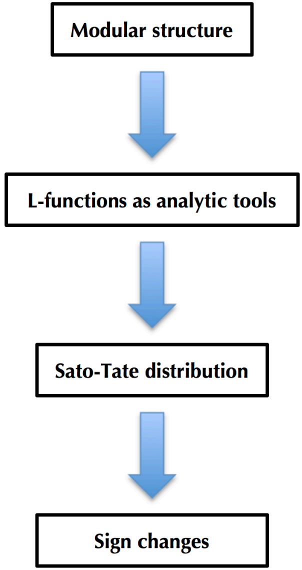
Elliptic curves

Theorem (Hasse–Weil conjecture, Wiles & BCDT)

The Hasse–Weil zeta function can be extended to a meromorphic function for all $s \in \mathbf{C}$, and should satisfy a functional equation.

1. This follows from the Shimura–Taniyama–Weil conjecture (Wiles, Breuil–Conrad–Diamond–Taylor) that the Hasse–Weil zeta function corresponds to the L-function of some cusp form f of weight 2: $a_p(E) = \lambda_f(p)$: the p-th Fourier coefficient of f for all "good" primes p.
2. The Sato–Tate conjecture for elliptic curves was proven by Clozel–Harris–Taylor based on their deep study on symmetric power L-functions.

Back to Kloosterman sums



Sign Changes

Towards I: sign changes

Theorem (Kuznetsov, Math. Sbornik [1980])

$$\sum_{n\leqslant X}\mathrm{Kl}(1,n) \ll X^{\frac{2}{3}+\varepsilon}.$$

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Theorem (Fouvry–Michel, Pacific J. Math. [2003])

$$\sum_{n\leqslant X} |\,\mathrm{Kl}(1,n)| \gg X \cdot \exp((\log\log X)^{-7/12}).$$

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- Fouvry–Michel used vertical Sato–Tate distribution for Kloosterman sums.
- Combining the works of Kuznetsov and Fouvry–Michel, Kloosterman sums change signs infinitely at consecutive integers.

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- Combining the works of Kuznetsov and Fouvry–Michel, Kloosterman sums change signs infinitely at consecutive integers.
- How about the case when the moduli runs over sparse integers? Primes?

Towards I: sign changes

Theorem (Fouvry–Michel, Ann. of Math. [2007])

$$\#\{X < n \leqslant 2X : \mathrm{Kl}(1,n) > 0, \mu^2(n) = 1, \omega(n) \leqslant 23\} \gg \frac{X}{\log X},$$
$$\#\{X < n \leqslant 2X : \mathrm{Kl}(1,n) < 0, \mu^2(n) = 1, \omega(n) \leqslant 23\} \gg \frac{X}{\log X}.$$

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The subsequent improvements are given by

- Sivak (18/2009)
- Matomäki (15/2011)
- Xi (10/2015, 7/2018)
- Drappeau–Maynard (2/2019, assuming the existence of Siegel zeros)

All the works combine sieve methods, ℓ-adic cohomology and spectral theory of automorphic forms.

Towards I: sign changes

Theorem (Xi, IMRN [2018])

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Theorem (Drappeau–Maynard, Proc. Amer. Math. Soc. [2019])

Let $\varepsilon > 0$. Then there are constants $A, B > 0$, depending only on ε, such that for $D > 2, X > D^A$ and any primitive real character $\chi \pmod D$, we have

$$\frac{1}{\pi(X)} \sum_{p \leqslant X} \mathrm{Kl}(1,p) \ll \varepsilon + B \cdot L(1,\chi_D) \log X.$$

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$$\frac{1}{\pi(X)} \sum_{p \leqslant X} \mathrm{Kl}(1,p) \ll \varepsilon + B \cdot L(1, \chi_D) \log X.$$

- Their method applies to moduli with more prime factors, and they need a lower bound for averages of $|\mathrm{Kl}(1,pq)|$ by Michel (Invent. math., 1995) or Fouvry–Kowalski–Michel (Duke Math. J., 2014).

Equidistributions

Towards II: equidistributions

The vertical analogue is true, although the horizontal one is not valid.

Theorem (Katz, Vertical Sato-Tate Theorem [1988])

As $p \to +\infty$, the set $\{\mathrm{Kl}(a,p) : a \in \mathbf{F}_p^\times\}$ becomes equidistributed in $[-2,2]$ w.r.t. the Sato–Tate measure $\frac{1}{2\pi}\sqrt{4-x^2}\mathrm{d}x$.

In other words, one has, for any interval $I \in [-2,2]$,

$$\lim_{p\to+\infty}\frac{1}{p-1}\#\{a \in \mathbf{F}_p^\times : \mathrm{Kl}(a,p) \in I\} = \frac{1}{2\pi}\int_I \sqrt{4-x^2}\mathrm{d}x.$$

Towards II: equidistributions (cont.)

For odd $c \geqslant 3$, define the *Salié sum* (a twited Kloosterman type sum) by

$$T(m,n;c) = \frac{1}{\sqrt{c}} \sum_{x \,(\mathrm{mod}\, c)} \left(\frac{x}{c}\right) \mathbf{e}\left(\frac{mx + n\bar{x}}{c}\right).$$

Theorem (Hooley, Acta Math. [1963])

The normalized Salié sums $T(1,1;c)$ equidistribute in $[-2,2]$ w.r.t. the Lebesgue measure $\mathrm{d}x$ as c runs over all positive odd integers.

Theorem (Duke–Friedlander–Iwaniec, Ann. of Math. [1995])

The normalized Salié sums $T(1,1;p)$ equidistribute in $[-2,2]$ w.r.t. the Lebesgue measure $\mathrm{d}x$ as p runs over all odd primes.

For $S = 2\mathbf{Z}^+ + 1$ or $\mathcal{P}$, one has

$$\lim_{X\to+\infty} \frac{1}{\#(S\cap[1,X])} \#\{c \in S \cap [1,X] : T(1,1;c) \in [\alpha,\beta]\} = \beta - \alpha.$$

Towards II: equidistributions (cont.)

Michel also studied the horizontal distribution on average.

Theorem (Michel, Duke Math. J. [1998])

Let $M = X^{1/2+\delta}$ *for some* $\delta > 0$. *Then there exists certain* $\eta = \eta(\delta) > 0$ *such that*

$$\frac{1}{M}\sum_{m\leqslant M}\left|\sum_{p\leqslant P}\mathrm{Kl}(m,p)\right| \ll P^{1-\eta}.$$

Remark: He used Heath-Brown Identity and Kloostermania of Deshouillers & Iwaniec. It is highly desired to study the general case

$$\frac{1}{M}\sum_{m\leqslant M}\left|\sum_{p\leqslant P,\ p\nmid m}\mathrm{sym}_k(\theta_p(m))\right|, \quad k \geqslant 1.$$

Towards II: equidistributions (cont.)

Although the original horizontal Sato–Tate distribution is not valid, Chai–Li proved a function field analogue.

Let K a function field over $\mathbf{F}_q$ with $\mathrm{char}(\mathbf{F}_q) = p$ and $\mathbf{F}_v$ the residue field of the completion of K at a place v.

$$\mathrm{Kl}(\mathbf{F}_v; a) = \frac{1}{\sqrt{|\mathbf{F}_v|}} \sum_{x \in \mathbf{F}_v^\times} \psi(\mathrm{Tr}_{\mathbf{F}_v/\mathbf{F}_q}(x + a/x)).$$

Theorem (Chai–Li, Forum Math. [2003])

Suppose a is not a constant in K. The normalized Kloosterman sums $\mathrm{Kl}(\mathbf{F}_v; a)$ equidistribute in $[-2, 2]$ w.r.t. the Sato–Tate measure as v varies through all places of K not occurring in a.

Remark: In fact, Chai–Li established the modular structures of the function field Kloosterman sums in the sense of Problem III of Katz.

Towards II: equidistributions (cont.)

For each $h \in \mathcal{C}^\infty([-2,2])$, it is expected that (by the Sato–Tate conjecture)

$$\varDelta_h(a,x) := \sum_{p \leqslant x} h(\mathrm{Kl}(a,p)) - \pi(x) \cdot T(h)$$

is *small*, where

$$T(h) = \frac{1}{2\pi} \int_{-2}^{2} h(t)\sqrt{4-t^2}\mathrm{d}t.$$

We may prove the following *Horizontal Central Limit Theorem*.

Theorem (Xi, Ramanujan J. [2017])

Let $h \in \mathcal{C}^\infty([-2,2])$ be a fixed non-constant function. As a runs over N consecutive integers, the random variables

$$a \mapsto \frac{\varDelta_h(a,x)}{\sqrt{\pi(x) \cdot (T(h^2) - T(h)^2)}}$$

converge in distribution to a standard Gaussian with mean 0 and variance 1 as $x, N \to +\infty$ with $\log x = o(\log N)$.

Modular Structures

Towards III: modular structures

Let f be a primitive Hecke–Maass cusp form of level $\mathfrak{q}$, trivial nebentypus and eigenvalue $\lambda = 1/4 + t^2$, so that it is a joint eigenfunction of the Laplacian and Hecke operators. Suppose f admits the following Fourier expansion

$$f(z) = \sqrt{y} \sum_{n \neq 0} \lambda_f(n) K_{it}(2\pi |n| y) \mathbf{e}(nx),$$

where $\lambda_f(1) = 1$ and K_ν is the K-Bessel function of order ν.

Conjecture (Ramanujan–Petersson Conjecture)

$$|\lambda_f(n)| \leqslant n^{\vartheta} \tau(n) \tag{1}$$

with $\vartheta = 0$.

Current record: $\vartheta = 7/64$ (Kim–Sarnak, 2003)

Towards III: modular structures (cont.)

Problem III of Katz is thus two-fold:

- $\lambda_f(n)$ is suggested to be controlled by virtue of Kloosterman sums.
- Spectral theory of Maass forms might be helpful to understand *non-trivial* analytic information about the Euler product $L_a(s)$, which would yield non-trivial progresses towards to Problems I and II.

But, too optimistic to be true?

Theorem (Booker, Exp. Math. [2000])

If $\mathrm{Kl}(1,p) = \pm\lambda_f(p)$ *for some primitive Hecke–Maass cusp form* f *of level* $\mathfrak{q} = 2^\nu$ *and eigenvalue* λ, *then*

$$(\lambda + 3) \cdot \mathfrak{q} > 18.3 \times 10^6.$$

This seems to be the only known result, based on numerical computations.

Towards III: modular structures (cont.)

Surprisingly, the function field analogue is really optimistic.

Theorem (Chai–Li, Forum Math. [2003])

Let K a function field with the field of constants $\mathbf{F}$ a finite field with $\mathrm{char}(\mathbf{F}) = p$*. Given $a \in K^\times$, there exists an automorphic form f of GL_2 over K which is an eigenfunction of the Hecke operator T_v at all places v of K, which is neither a zero nor a pole of a, with eigenvalue*

$$-\mathrm{Kl}(\mathbf{F}_v; a) = \frac{-1}{\sqrt{|\mathbf{F}_v|}} \sum_{x \in \mathbf{F}_v^\times} \psi(\mathrm{Tr}_{\mathbf{F}_v/\mathbf{F}_q}(x + a/x)).$$

Here $\mathbf{F}_v$ denotes the residue field of the completion of K at v. In other words,

$$L(s,f) \sim \prod_{v \text{ good}} \left(1 + \frac{\mathrm{Kl}(\mathbf{F}_v; a)}{\mathrm{N}(v)^s} + \frac{1}{\mathrm{N}(v)^{2s}}\right)^{-1}.$$

Towards III: modular structures (cont.)

Towards the original problem of Katz, we have the following approximation.

Theorem (Xi, Invent. math. [2020])

For each primitive Hecke–Maass cusp form f of trivial nebentypus, there exists infinitely many squarefree n with at most 100 prime factors, such that

$$\lambda_f(n) \neq \pm \mathrm{Kl}(1, n).$$

Towards III: modular structures (cont.)

In fact, we can prove the following general version.

Theorem (Xi, Invent. math. [2020])

Let $\eta \in \mathbf{R}$. For each primitive Hecke–Maass cusp form f of trivial nebentypus, there exist some constant $r = r(\eta) < +\infty$, such that

$$\#\{X < n \leqslant 2X : \mathrm{Kl}(1, n) > \eta \cdot \lambda_f(n),\ \omega(n) \leqslant r,\ \mu^2(n) = 1\} \gg \frac{X}{\log X},$$

$$\#\{X < n \leqslant 2X : \mathrm{Kl}(1, n) < \eta \cdot \lambda_f(n),\ \omega(n) \leqslant r,\ \mu^2(n) = 1\} \gg \frac{X}{\log X}.$$

In particular, one may take $r(\pm 1) = 100$, $r(\pm\frac{1}{2018}) = 41$, $r(\pm 2018) = 25$.

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- This theorem is completely new, even if there is no restriction on the size of $\omega(n)$.
- The merit of this theorem is revealed by the flexibility of η.

Approaching horizontal

- Many of the above results use a lot of analytic number theory, especially sieve methods and automorphic forms. Sieve method allows one to transfer from almost primes to consecutive integers, and sums of Kloosterman sums over integers can be interpreted by spectral theory of automorphic forms.
- In addition, tools from ℓ-adic cohomology play a crucial role after transferring from horizontal to vertical via the following twisted multiplicativity:

$$\mathrm{Kl}(m, rs) = \mathrm{Kl}(m\bar{s}^2, r)\,\mathrm{Kl}(m\bar{r}^2, s), \quad (r, s) = 1.$$

- These would require certain estimates for

$$\sum_{n \in \mathcal{S}} \alpha_n \,\mathrm{sym}_k(\theta_p(n))$$

for different shapes of $\mathcal{S}$.

Sieving: a weighted Selberg sieve

- Suppose $(a_n)_{n\leqslant x}$ is a sequence of non-negative numbers. Sieve methods were originally designed to capture how often these numbers are supported on primes, although current status only allows us to detect *almost primes* in most cases.

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- A convenient approach was invented by Selberg in 1950's in connection with the twin prime conjecture and Goldbach problem. Precisely, he suggests to consider

$$\sum_{x<n\leqslant 2x} a_n w_n\{\rho-\tau(n)\},$$

where w_n is a non-negative function, and ρ is to be chosen appropriately such that the total average is positive for all sufficiently large x, from which one obtains the existence of n such that $\omega(n)\leqslant \log\rho/\log 2$. The ingenuity then lies in the choice of w_n.

Sieving: a weighted Selberg sieve (cont.)

$$H^{\pm}(X)=\sum_{X<n\leqslant 2X}^{\flat}\{|\psi(n)|\pm\psi(n)\}\{\rho-\tau_{\Delta}(n;\alpha,\beta)\}\Big(\sum_{d|(n,\Pi_\theta)}\varrho_d\Big)^2.$$

- $$\psi(n):=\mathrm{Kl}(1,n)-\eta\cdot\lambda_f(n).$$
- $$\tau_{\Delta}(n;\alpha,\beta)=\sum_{\substack{dl=n\\ d\leqslant l^{\Delta}}}\alpha^{\omega(d)}\beta^{\omega(l)},\quad \alpha,\beta,\Delta>0.$$
- $$\varrho_d=\mu(d)\Big(\frac{\log(\sqrt{D}/d)}{\log\sqrt{D}}\Big)^2\mathbf{1}_{d\leqslant X^{\delta}},\quad \Pi_\theta=\prod_{p<X^\theta}p.$$

Sieving (cont.)

$$\sum_{\substack{p_1,p_2,\cdots,p_k \\ n=p_1p_2\cdots p_k}} |\,\mathrm{Kl}(1,n)-\eta\cdot\lambda_f(n)\,|^{\ell}, \quad (\ell=1,2,4).$$

- $\|\boldsymbol{\alpha}\|_1 \geqslant \|\boldsymbol{\alpha}\|_2^3/\|\boldsymbol{\alpha}\|_4^2$.
- Lower bound for

$$\sum_{\substack{p_1,p_2,\cdots,p_k \\ n=p_1p_2\cdots p_k}} (|\,\mathrm{Kl}(1,n)\,|^2+|\,\lambda_f(n)\,|^2).$$

- Upper bound for (very crucially)

$$\sum_{\substack{p_1,p_2,\cdots,p_k \\ n=p_1p_2\cdots p_k}} |\,\mathrm{Kl}(1,n)\lambda_f(n)\,|.$$

- We need **equidistributions** of Kloosterman sums in the vertical direction.

Equidistributions

Following Deligne and Katz, it is known that $a \in \mathbf{F}_p^\times \mapsto -\mathrm{Kl}(a,p)$ is the trace function of an ℓ-adic sheaf $\mathcal{K}\ell$ on $\mathbf{G}_m(\mathbf{F}_p) = \mathbf{F}_p^\times$, which is of rank 2 and pure of weight 0. Alternatively, we may write

$$\cos\theta_p(a) =: \mathrm{Kl}(a,p) = \mathrm{tr}(\mathrm{Frob}_a, \mathcal{K}\ell), \quad a \in \mathbf{F}_p^\times \text{ and } \theta_p(a) \in [0,\pi].$$

Lemma (Weyl's criterion)

Let G be a compact group and $G^\sharp$ the set of conjugacy classes in G. Then $\{x_n\} \subset G^\sharp$ is equidistributed with respect to a Haar measure μ if and only if for any non-trivial irreducible unitary representation $\rho : G \to GL(V)$, we have

$$\sum_{n \leqslant N} \mathrm{tr}(\rho(x_n)) = o(N).$$

- $G = [0,\pi]$, $\mu = \frac{2}{\pi}\sin^2\theta \mathrm{d}\theta$ (Sato–Tate), $\mathrm{tr}(\rho_k) = \mathrm{sym}_k(\theta) = \frac{\sin((k+1)\theta)}{\sin\theta}$.

Equidistributions

[Grothendieck–Lefschetz trace formula]

$$\sum_{a\in\mathbf{F}_p^\times}\mathrm{sym}_k(\theta_p(a))=\sum_{a\in\mathbf{F}_p^\times}\mathrm{tr}(\mathrm{Frob}_a,\mathrm{sym}^k\mathcal{K}\ell)$$
$$=\sum_{i=0}^{2}(-1)^i\,\mathrm{tr}(\mathrm{Frob}_p\,|H_c^i(U\times\overline{\mathbf{F}}_p,\mathrm{sym}^k\mathcal{K}\ell)),$$

where $\mathrm{sym}^k\mathcal{K}\ell$ is the k-th symmetric power of the Kloosterman sheaf $\mathcal{K}\ell$.

Theorem (Katz, 1988)

$$\left|\sum_{a\in\mathbf{F}_p^\times}\mathrm{sym}_k(\theta_p(a))\right|\ll(k+1)\sqrt{p}.$$

Equidistributions (cont.)

Lemma (Michel, Invent. math. [1995])

Let p be a large prime and $(a,p)=1$. For each $k \geqslant 1$ and any coefficients $\boldsymbol{\alpha}=(\alpha_m), \boldsymbol{\beta}=(\beta_n)$, we have

$$\sum_{\substack{m\sim M\\(mn,p)=1}}\sum_{n\sim N}\alpha_m\beta_n \mathrm{sym}_k(\theta_p(a\overline{(mn)^2}))$$

$$\ll \|\boldsymbol{\alpha}\|\|\boldsymbol{\beta}\|(MN)^{\frac12}(p^{-\frac14}+N^{-\frac12}+M^{-\frac12}p^{\frac14}(\log p)^{\frac12}),$$

where the implied constant depends polynomially on k.

- Non-trivial for $M > p^{\frac12}(\log p)^2$ and $N > \log p$.
- The sheaves $\mathrm{sym}_k([-2]^*\mathcal{K}\ell)$ and $\mathrm{sym}_k(([-2]^*\mathcal{K}\ell_a)$ are geometrically irreducible and not geometrically isomorphic for $a \neq 0, 1$.

Lemma (Fouvry–Michel, Ann. of Math. [2007])

Suppose $q = q_1 q_2 \cdots q_s$ *with* $q_1, q_2, \cdots, q_s$ *being distinct primes and* $(a, q) = 1$. *For each* s*-tuple of positive integers* $\mathbf{k} = (k_1, k_2, \cdots, k_s)$, *and any coefficients* $\boldsymbol{\alpha} = (\alpha_m), \boldsymbol{\beta} = (\beta_n), \boldsymbol{\gamma} = (\gamma_{m,n})$ *with* $m \equiv m' \pmod n \Rightarrow \gamma_{m,n} = \gamma_{m',n}$, *we have*

$$\sum_{\substack{m\sim M \\ (mn,q)=1}} \sum_{n\sim N} \alpha_m \beta_n \gamma_{m,n} \prod_{1\leqslant j\leqslant s} \mathrm{sym}_{k_j}(\theta_{q_j}(a\overline{(mnq/q_j)^2}))$$

$$\ll c(s;\mathbf{k})\|\boldsymbol{\alpha}\|\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|_\infty (MN)^{\frac{1}{2}}(q^{-\frac{1}{8}} + N^{-\frac{1}{4}}q^{\frac{1}{8}} + M^{-\frac{1}{2}}N^{\frac{1}{2}}),$$

where $c(s;\mathbf{k}) = 3^s \prod_{j=1}^s (k_j + 1)$ *and the implied constant is absolute.*

- Non-trivial for $N > q^{\frac{1}{2}} \log q$ and $M > N \log q$.

Equidistributions (cont.)

Lemma

Let $P, M \geqslant 3$ and $a \neq 0$. Suppose $\boldsymbol{\gamma} = (\gamma_p)$ is a complex coefficient supported on primes in $]P, 2P]$. Then there exisits some absolute constant $A > 0$, such that for each $k \geqslant 1$ and arbitrary coefficient $\boldsymbol{\alpha} = (\alpha_m)$ supported in $]M, 2M]$,

$$\sum_{\substack{p \sim P \\ (p,a)=1}} \gamma_p \sum_{m \sim M} \alpha_m \mathrm{sym}_k(\theta_p(a\overline{m^2})) \ll k^A (M^{\frac{1}{2}} + P \log P) \|\boldsymbol{\alpha}\| \|\boldsymbol{\gamma}\|$$

holds with some implied constant depending at most on A.

- Non-trivial for $M > P(\log P)^2$.

Evans conjecture for moments of Kloosterman sums

Evans has conjectured the modularity of the following moments:

$$\mathcal{M}_k(p) = \sum_{a \in \mathbf{F}_p^\times} \mathrm{sym}_k(\theta_p(a)).$$

- (Peters–Top –van der Vlugt, 1992; Livné, 1995)
  $\sqrt{p}\mathcal{M}_5(p)$ is the p-th Fourier coefficient of a holomorphic cuspdial Hecke eigenform of weight 3, level 15 and quadratic nebentypus $(\cdot/15)$.
- (Hulek–Spandaw–van Geemen–van Straten, 2001)
  $p\mathcal{M}_6(p)$ is *almost* the p-th Fourier coefficient of a holomorphic cuspdial Hecke eigenform of weight 4 and level 6.
- (Yun, 2015)
  $p^{3/2}\mathcal{M}_7(p)$ is *almost* the square of the p-th Fourier coefficient of a holomorphic cuspdial Hecke eigenform of weight 3 and level 525 and nebentypus $(\cdot/21)\chi_5$.
  $p^2\mathcal{M}_8(p)$ is *almost* the square of the p-th Fourier coefficient of a holomorphic cuspdial Hecke eigenform of weight 6 and level 6.

Remarks

- With the input from algebraic geometry, Kloosterman sums, as well as many other algebraic exponential sums, may provide powerful tools in modern analytic number theory.
- Classical methods in analytic number theory can also be employed to demonstrate certain objects in arithmetic geometry.

Thank you for your attention !

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