Beilinson–Bloch conjecture and arithmetic inner product formula

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MCM & YMSC Number Theory Seminar

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Put $G_r := \mathrm{U}(W_r)$, the unitary group of W_r, which is a quasi-split reductive group over F.

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Then we have $\varepsilon(\pi) = (-1)^{|\mathtt{S}_\pi|}$ for the global (doubling) root number, so that the vanishing order of $L(s,\pi)$ at the center $s=\frac{1}{2}$ has the same parity as $|\mathtt{S}_\pi|$.

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Then we have $\varepsilon(\pi) = (-1)^{|\mathtt{S}_\pi|}$ for the global (doubling) root number, so that the vanishing order of $L(s,\pi)$ at the center $s = \frac{1}{2}$ has the same parity as $|\mathtt{S}_\pi|$.
The cuspidal automorphic representation π determines a hermitian space V_π over $\mathbb{A}_E$ of rank n via local theta dichotomy (so that the local theta lifting of π_v to $\mathrm{U}(V_\pi)(F_v)$ is nontrivial for every place v of F), unique up to isomorphism, which is totally positive definite and satisfies that for every $v \in \mathtt{V}_F^{\mathrm{fin}}$, the local Hasse invariant $\epsilon(V_\pi \otimes_{\mathbb{A}_F} F_v) = 1$ if and only if $v \notin \mathtt{S}_\pi$.

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Let $\mathtt{R}$ be a finite subset of $\mathtt{V}_F^{\mathrm{fin}}$. We fix a special maximal subgroup $L^{\mathtt{R}}$ of $H(\mathbb{A}_F^{\infty,\mathtt{R}})$ that is the stabilizer of a lattice $\Lambda^{\mathtt{R}}$ in $V \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\infty,\mathtt{R}}$ such that $\Lambda_v^{\mathtt{R}}$ is self-dual (resp. almost self-dual) for $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{S}_\pi$ (resp. $v \in \mathtt{S}_\pi$).

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For a field $\mathbb{L}$, we denote by $\mathbb{T}^{\mathtt{R}}_{\mathbb{L}}$ the (abstract) Hecke algebra $\mathbb{L}[L^{\mathtt{R}}\backslash H(\mathbb{A}_F^{\infty,\mathtt{R}})/L^{\mathtt{R}}]$, which is a commutative $\mathbb{L}$-algebra.

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When $\mathtt{R}$ contains $\mathtt{R}_\pi$, the cuspidal automorphic representation π gives rise to a character

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Let π be as in the Setup with $|\mathtt{S}_\pi|$ odd. If $L'(\frac{1}{2}, \pi) \neq 0$, that is, $\mathrm{ord}_{s=\frac{1}{2}} L(s,\pi) = 1$, then as long as $\mathtt{R}$ satisfies $\mathtt{R}_\pi \subseteq \mathtt{R}$ and $|\mathtt{R} \cap \mathtt{V}_F^{\mathrm{spl}}| \geq 2$, the nonvanishing

$$\varinjlim_{L_{\mathtt{R}}} \left(\mathrm{CH}^r(X_{L_{\mathtt{R}} L^{\mathtt{R}}})^0_{\mathbb{Q}^{\mathrm{ac}}} \right)_{\mathfrak{m}_\pi^{\mathtt{R}}} \neq \{0\}$$

holds, where the colimit is taken over all open compact subgroups $L_{\mathtt{R}}$ of $H(F_{\mathtt{R}})$.

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We also have a refined version of the above theorem, in which we give an explicit height formula for certain cycles in $\mathrm{CH}^r(X_L)^0_{\mathbb{C}}$. However, the refined version is conditional on a hypothesis on the **modularity of Kudla's generating functions**.

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Now we recall Kudla's special cycles and generating functions. Take an open compact subgroup $L \subseteq H(\mathbb{A}_F^\infty)$ and a test function $\phi^\infty \in \mathscr{S}(V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^L$.

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For every $T \in \mathrm{Herm}_r(F)^+$, we have an element $Z_T(\phi^\infty)_L \in \mathrm{CH}^r(X_L)_{\mathbb{C}}$, which is the ϕ^∞-weighted (finite) linear combination of **special cycles** indexed by $x \in V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty$ with $T(x) = T$.

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Then Kudla's **generating function**, which is a function on $g \in G_r(\mathbb{A}_F)$, is defined as

$$Z_{\phi^\infty}(g)_L \coloneqq \sum_{T \in \mathrm{Herm}_m(F)^+} (\omega_{r,\infty}(g_\infty)\phi_\infty^0)(T) \cdot Z_T(\omega_r^\infty(g^\infty)\phi^\infty)_L$$

as a formal series valued in $\mathrm{CH}^r(X_L)_\mathbb{C}$.

Arithmetic theta lifting

Hypothesis (Modularity of generating functions)

For every L and ϕ^∞ as above, and every complex linear map $l\colon \mathrm{CH}^r(X_L)_\mathbb{C} \to \mathbb{C}$, the assignment

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Definition

Assuming the Modularity Hypothesis, for every L and ϕ^∞ as above and a holomorphic vector $\varphi \in \pi$, we define the **arithmetic theta lifting** to be

$$\Theta_{\phi^\infty}(\varphi)_L \coloneqq \int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \overline{\varphi(g)} Z_{\phi^\infty}(g)_L \mathrm{d}g$$

which is an element in $\mathrm{CH}^r(X_L)_{\mathbb{C}}$.

Main results, cont.

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Take L and two pairs $(\phi_1^\infty, \varphi_1)$ and $(\phi_2^\infty, \varphi_2)$ as in the previous definition, we have a natural height pairing

$$\langle \Theta_{\phi_1^\infty}(\varphi_1)_L, \Theta_{\phi_2^\infty}(\varphi_2)_L \rangle_{X_L,E}$$

as a restricted form of Beilinson's (hermitian) height pairing.

Main results, cont.

Take L and two pairs $(\phi_1^\infty, \varphi_1)$ and $(\phi_2^\infty, \varphi_2)$ as in the previous definition, we have a natural height pairing

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Let π be as in the Setup with $|\mathtt{S}_\pi|$ odd. Assume the Modularity Hypothesis.

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- ✓ $\varphi_1 = \otimes_v \varphi_{1v} \in \pi$ *and* $\varphi_2 = \otimes_v \varphi_{2v} \in \pi$ *that are holomorphic such that* $\langle \overline{\varphi_{1v}}, \varphi_{2v} \rangle_{\pi_v} = 1$ *for every* $v \in \mathtt{V}_F^{(\infty)}$,
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the ***arithmetic inner product formula***

$$\langle \Theta_{\phi_1^\infty}(\varphi_1), \Theta_{\phi_2^\infty}(\varphi_2) \rangle_{X,E}^\natural = \frac{L'(\frac{1}{2}, \pi)}{b_{2r}(0)} \cdot C_r^{[F:\mathbb{Q}]} \cdot \prod_{v \in \mathtt{V}_F^{\mathrm{fin}}} \mathfrak{Z}_{\pi_v, V_v}^\natural(\overline{\varphi_{1v}}, \varphi_{2v}, \phi_{1v}^\infty \otimes \overline{\phi_{2v}^\infty})$$

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The arithmetic inner product formula is perfectly parallel to the classical Rallis inner product formula. In fact, suppose that V is totally positive definite but *coherent*. We have the classical theta lifting $\theta_{\phi^\infty}(\varphi)$ where we use Gaussian functions at archimedean places. Then the Rallis inner product formula in this case reads as

$$\langle \theta_{\phi_1^\infty}(\varphi_1), \theta_{\phi_2^\infty}(\varphi_2) \rangle_H = \frac{L(\frac{1}{2}, \pi)}{b_{2r}(0)} \cdot C_r^{[F:\mathbb{Q}]} \cdot \prod_{v \in \mathtt{V}_F^{\mathrm{fin}}} \mathfrak{Z}^\natural_{\pi_v,V_v}(\overline{\varphi_{1v}}, \varphi_{2v}, \phi^\infty_{1v} \otimes \overline{\phi^\infty_{2v}}),$$

in which $\langle\ ,\ \rangle_H$ denotes the Petersson inner product with respect to the *Tamagawa measure* on $H(\mathbb{A}_F)$.

Main results, cont.

Corollary

Let π be as in the Setup with $|\mathrm{S}_\pi|$ odd. Assume the Modularity Hypothesis. In the context of the Beilinson–Bloch conjecture, take ($V = V_\pi$ and) $\tilde{\pi}^\infty$ to be the theta lifting of π^∞ to $H(\mathbb{A}_F^\infty)$.

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$$\mathrm{Hom}_{H(\mathbb{A}_F^\infty)}\left(\tilde{\pi}^\infty, \varinjlim_{L} \mathrm{CH}^r(X_L)^0_{\mathbb{C}}\right) \neq \{0\}.$$

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In the situation of the refined Theorem, suppose further that

- ✓ $\mathtt{R}_\pi = \emptyset$;
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Remark

Beilinson conjectures that $(-1)^r\langle\Theta_{\phi^\infty}(\varphi),\Theta_{\phi^\infty}(\varphi)\rangle^\natural_{X,E}\geq 0$, which, in the situation of the previous Corollary, is equivalent to $L'(\frac{1}{2},\pi)\geq 0$.

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which is defined as follows: It suffices to define for two algebraic cycles (with integral coefficients) (Z_1,Z_2) such that $|Z_1|\cap|Z_2|=\emptyset$. Then

$$\langle Z_1,Z_2\rangle^\ell_{X,E}=\sum_u c(u)\langle Z_1,Z_2\rangle^\ell_{X_u,E_u}$$

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  Then $\langle Z_1, Z_2\rangle^{\ell}_{X_u,E_u}$ equals the image of $\gamma_1 \cup \gamma_2$ under the composite map

$$\mathrm{H}^{2N}(U_1 \cap U_2, \mathbb{Q}_\ell(N+1)) \to \mathrm{H}^{2N+1}(X_u, \mathbb{Q}_\ell(N+1)) \xrightarrow{\mathrm{Tr}_{X_u}} \mathrm{H}^1(\mathrm{Spec}\, E_u, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell.$$

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- ✓ when u is archimedean, we have

$$\langle Z_1, Z_2\rangle^{\ell}_{X_u,E_u} = \frac{1}{2}\int_{X_u(\mathbb{C})} g_{Z_1} \wedge \delta_{Z_2} \in \mathbb{C}$$

  where g_{Z_1} is a harmonic Green current for Z_1 and δ_{Z_2} denotes the Dirac current of Z_2;
- ✓ when u is nonarchimedean, we define $\langle Z_1, Z_2\rangle^{\ell}_{X_u,E_u} \in \mathbb{Q}_\ell$ as follows:
  Let $c_i \in \mathrm{H}^{2d_i}_{|Z_i|}(X_u, \mathbb{Q}_\ell(d_i))$ be the refined cycle class of Z_i.
  As c_i goes to zero in $\mathrm{H}^{2d_i}(X_u, \mathbb{Q}_\ell(d_i))$, there exists $\gamma_i \in \mathrm{H}^{2d_i-1}(U_i, \mathbb{Q}_\ell(d_i))$ that goes to c_i under the coboundary map $\mathrm{H}^{2d_i-1}(U_i, \mathbb{Q}_\ell(d_i)) \to \mathrm{H}^{2d_i}_{|Z_i|}(X_u, \mathbb{Q}_\ell(d_i))$, where $U_i := X_u \setminus |Z_i|$.
  Then $\langle Z_1, Z_2\rangle^{\ell}_{X_u,E_u}$ equals the image of $\gamma_1 \cup \gamma_2$ under the composite map

$$\mathrm{H}^{2N}(U_1 \cap U_2, \mathbb{Q}_\ell(N+1)) \to \mathrm{H}^{2N+1}(X_u, \mathbb{Q}_\ell(N+1)) \xrightarrow{\mathrm{Tr}_{X_u}} \mathrm{H}^1(\mathrm{Spec}\, E_u, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell.$$

Conjecturally, one should have $\mathrm{CH}^d(X)^{\langle \ell \rangle} = \mathrm{CH}^d(X)^0$, and that $\langle\ ,\ \rangle^{\ell}_{X,E}$ takes values in $\mathbb{C}$ and is independent of ℓ.

Thank you!