Online Number Theory Seminar

Connectedness of Kisin varieties associated to absolutely irreducible Galois representations

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(joint with Sian Nie)

Notations

- K: a finite extension of $\mathbb{Q}_p$ with $p > 2$
- $\Gamma_K := \mathrm{Gal}(\bar{K}|K)$ absolute Galois group of K
- $\mathbb{F}$: a finite field of characteristic p
- $\bar{\rho} : \Gamma_K \to \mathrm{GL}_n(\mathbb{F})$, a n-dimensional continuous representation of Γ_K.
- $\mathrm{Rep}_{\mathbb{F}_p}(\Gamma_K)$: category of finite dimensional continuous $\mathbb{F}_p$-representations of Γ_K.
- FGp_K: category of (commutative) finite group schemes over K which are p-torsion.
- There is a natural equivalence of categories:

$$\mathrm{Rep}_{\mathbb{F}_p}(\Gamma_K) \simeq \mathrm{FGp}_K.$$

In particular, $\bar{\rho}$ can be viewed as a finite group scheme over K.

Kisin varieties

Kisin constructed a projective scheme $C(\bar{\rho})$ over $\mathbb{F}$ such that for any $\mathbb{F}'|\mathbb{F}$,

$$C(\bar{\rho})(\mathbb{F}') := \left\{ \begin{array}{c} \text{finite flat group schemes over } \mathcal{O}_K \\ \text{with generic fiber } \bar{\rho} \otimes_{\mathbb{F}} \mathbb{F}' \end{array} \right\}.$$

Variant : Kisin variety with tame descent datum.
When the ramfication index $e(K|\mathbb{Q}_p) < p - 1$, Raynaud showed that $C(\bar{\rho})$ has at most one point.

Motivation:

- modularity lifting
- Breuil-Mézard conjecture (variant version)

Kisin varieties $\leftrightsquigarrow$ Local deformation ring

$\bar{\rho}$ via p-adic Hodge theory

Recall $\bar{\rho} : \Gamma_K \to \mathrm{GL}_n(\mathbb{F})$.

- k: the residue field of K
- π be a uniformizer of K
- $\pi_n := \pi^{\frac{1}{p^n}}$ be a compatible system of p^n-th root of π for all $n \in \mathbb{N}$.
- $K_\infty := \cup_n K(\pi_n)$.

Theorem (Fontaine-Wintenberger)

There exists a canonical isomprhism of absolute Galois groups

$$\Gamma_{K_\infty} \simeq \Gamma_{k((u))}$$

where $\Gamma_{K_\infty} = \mathrm{Gal}(\bar{K}_\infty | K_\infty)$ *and* $\Gamma_{k((u))} = \mathrm{Gal}(k((u))^{sep} | k((u)))$.

$\bar{\rho}_{|\Gamma_{K_\infty}}$ can be viewed as a $\mathbb{F}$-representation of $\Gamma_{k((u))}$.

Classification of mod p-representations of $\Gamma_{k((u))}$

Let φ be the absolute Frobenius on $k((u))$.
An étale φ-module over $k((u))$ is a pair (N, Φ) where

- N is a $k((u))$-vector space of finite rank,
- $\Phi : N \to N$ is semi-linear with respect to φ,

such that $1 \otimes \Phi : \varphi^*(N) \to N$ is an isomorphism.

Let $\mathrm{Mod}^{\varphi,et}_{k((u))}$ be the category of étale φ-modules over $k((u))$.

Theorem (Fontaine)

There is an equivalence of categories

$$\mathrm{Rep}_{\mathbb{F}_p}(\Gamma_{k((u))}) \simeq \mathrm{Mod}^{\varphi,et}_{k((u))}.$$

Breuil-Kisin classification of finite flat group schemes over $\mathcal{O}_K$

- $\mathrm{FFGp}_{\mathcal{O}_K}$: the category of finite flat group schemes over $\mathcal{O}_K$ which is killed by p.
- $\mathrm{Mod}^{\varphi}_{k[\![u]\!]}$: the category of finite free $k[\![u]\!]$-modules $\mathfrak{M}$ with an injective semi-linear map $\Phi : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of $\varphi^*\mathfrak{M} \to \mathfrak{M}$ is killed by u^e, where $e = e(K|\mathbb{Q}_p)$.

Note that $\mathfrak{M}[\frac{1}{u}]$ is an étale φ-module over $k(\!(u)\!)$, for $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{k[\![u]\!]}$.

Breuil-Kisin classification of finite flat group schemes over $\mathcal{O}_K$

Theorem (Breuil-Kisin)

There is an equivalence of categories

$$\mathrm{FFGp}_{\mathcal{O}_K} \overset{BK}{\simeq} \mathrm{Mod}^{\varphi}_{k[\![u]\!]}$$
$$\mathcal{G} \mapsto \mathrm{BK}(\mathcal{G})$$

Moreover, for $\mathcal{G} \in \mathrm{FFGp}_{\mathcal{O}_K}$, then there is a canonical isomorphism of étale φ-modules over $k(\!(u)\!)$:

$$N_{\mathcal{G}_K(-1)} \simeq \mathrm{BK}(\mathcal{G})[\frac{1}{u}].$$

Summary

$$
\begin{array}{ccccc}
\bar{\rho} \in \mathrm{Rep}_{\mathbb{F}_p}(\Gamma_K) & \longrightarrow & \mathrm{Rep}_{\mathbb{F}_p}(\Gamma_{K_\infty}) & \xrightarrow{\sim} & \mathrm{Rep}_{\mathbb{F}_p}(\Gamma_{k((u))}) \\
\uparrow \wr & & & & \downarrow \wr \\
\mathrm{FGp}_K & & & & \mathrm{Mod}^{\varphi, et}_{k((u))} \\
\text{generic} \uparrow \text{fiber} & & & & \uparrow (-)[\frac{1}{u}](1) \\
\mathrm{FFGp}_{\mathcal{O}_K} & & \xrightarrow[\sim]{BK} & & \mathrm{Mod}^{\varphi}_{k[\![u]\!]}
\end{array}
$$

$\bar{\rho}(-1)_{|\Gamma_{K_\infty}}$ in terms of an étale φ-module

Recall: $\bar{\rho}_{|\Gamma_{K_\infty}} : \Gamma_{k((u))} \to \mathrm{GL}_n(\mathbb{F})$.

Let $\varphi : \mathbb{F} \otimes_{\mathbb{F}_p} k((u)) \overset{1\otimes\varphi}{\to} \mathbb{F} \otimes_{\mathbb{F}_p} k((u))$.

Hence $\bar{\rho}_{|\Gamma_{K_\infty}}(-1)$ can be viewed as an étale φ-module $N_{\bar{\rho}} = (N_{\bar{\rho}}, \Phi_{\bar{\rho}})$ with $\mathbb{F}$-action. More precisely,

- $N_{\bar{\rho}}$ is a free $\mathbb{F} \otimes_{\mathbb{F}_p} k((u))$-module of rank n;
- $\Phi_{\bar{\rho}} : N_{\bar{\rho}} \to N_{\bar{\rho}}$ is semi-linear with respect to φ such that $\varphi^* N_{\bar{\rho}} \to N_{\bar{\rho}}$ is an isomorphism.

Affine Grassmannian

Let $G = \mathrm{Res}_{k|\mathbb{F}_p}\mathrm{GL}_n$.

The affine Grassmannian for G is the ind-projective scheme over $\mathbb{F}_p$ such that for any $\mathbb{F}'|\mathbb{F}_p$:

$$\mathcal{G}rass_G(\mathbb{F}') = G(\mathbb{F}'(\!(u)\!))/G(\mathbb{F}'[\![u]\!]).$$

This parametrizes lattices inside $\mathbb{F}' \otimes_{\mathbb{F}_p} k(\!(u)\!)^n$.

Kisin variety in terms of moduli space of lattices

For any $\mathbb{F}'|\mathbb{F}$,

$$C(\bar{\rho})(\mathbb{F}') = \left\{ \begin{matrix} \mathfrak{M} \subset N_{\bar{\rho}} \otimes_{\mathbb{F}} \mathbb{F}' \text{ lattice such that} \\ u^e \mathfrak{M} \subset \Phi_{\bar{\rho}}(\varphi^* \mathfrak{M}) \subset \mathfrak{M} \end{matrix} \right\},$$

where $\Phi_{\bar{\rho}}(\varphi^* \mathfrak{M})$ denote the image of $1 \otimes \Phi_{\bar{\rho}} : \varphi^* \mathfrak{M} \to \mathfrak{M}$.

$\rightsquigarrow$ Kisin variety is a closed subscheme inside the affine Grassmannian $\mathcal{G}rass_G$.

More generally, we consider a closed subscheme $C_\mu(\bar{\rho})$ inside $C(\bar{\rho})$:

$$C_\mu(\bar{\rho})(\mathbb{F}') := \left\{ \begin{matrix} \mathfrak{M} \subset N_{\bar{\rho}} \otimes_{\mathbb{F}} \mathbb{F}' \text{ lattice such that the relative} \\ \text{position of } \Phi_{\bar{\rho}}(\varphi^* \mathfrak{M}) \text{ and } \mathfrak{M} \text{ is controlled by } \mu \end{matrix} \right\}$$

where μ is a cocharacter of G.

Relation of Kisin varieties and deformation rings

$R_{\bar\rho}^{fl,\nu}$: flat deformation ring of $\bar\rho$ such that Hodge-Tate weights are given by ν, which is a cocharacter of $\mathrm{Res}_{K|\mathbb{Q}_p}\mathrm{GL}_n$.

To ν, we can associate a cocharacter of G:
$\mu(\nu) : \mathbb{G}_{m,|\bar{\mathbb{F}}_p} \stackrel{\nu\otimes\bar{\mathbb{F}}_p}{\to} \mathrm{Res}_{\mathcal{O}_K|\mathbb{Z}_p}(\mathrm{GL}_n)_{|\bar{\mathbb{F}}_p} \to \mathrm{Res}_{k|\mathbb{F}_p}(\mathrm{GL}_n)_{|\bar{\mathbb{F}}_p}$.

Theorem (Kisin)

There is a bijection:

$$\pi_0(C_{\mu(\nu)}(\bar\rho)) \simeq \pi_0(\mathrm{Spec}(R_{\bar\rho}^{fl,\nu}[\frac{1}{p}])).$$

Question: $\pi_0(C_\mu(\bar\rho)) = ?$

Multiplicative and étale rank of finite flat groups over $\mathcal{O}_K$

- $\mathcal{G}$: a finite flat group scheme over $\mathcal{O}_K$;
- There is a connected-étale exact sequence:

$$0 \to \mathcal{G}^0 \to \mathcal{G} \to \mathcal{G}^{et} \to 0.$$

- Cartier dual $\mathcal{G}^\vee$: for any scheme T over $\mathcal{O}_K$,

$$\mathcal{G}^\vee(T) := \mathrm{Hom}_{T-gp}(\mathcal{G}_T, \mathbb{G}_{m,T}).$$

  We have $\mathcal{G}^{\vee\vee} = \mathcal{G}$.
- $\mathcal{G}^m := (\mathcal{G}^{\vee,et})^\vee \subset \mathcal{G}$ is the maximal multiplicative subgroup scheme of $\mathcal{G}$.

Multiplicative and étale rank of finite flat groups over $\mathcal{O}_K$

Define:

- $d_m(\mathcal{G}) := \mathrm{rank}\mathcal{G}^m$ multiplicative rank of $\mathcal{G}$,
- $d_{et}(\mathcal{G}) := \mathrm{rank}\mathcal{G}^{et}$ étale rank of G.

Suppose $(\mathfrak{M}, \Phi) = \mathrm{BK}(\mathcal{G})$.

- $\mathcal{G}$ is étale (i.e. $\mathcal{G} = \mathcal{G}^{et}$) if and only if $\Phi(\varphi^*\mathfrak{M}) = u^e\mathfrak{M}$;
- $\mathcal{G}$ is multiplicative (i.e. $\mathcal{G} = \mathcal{G}^m$) if and only if $\Phi(\varphi^*\mathfrak{M}) = \mathfrak{M}$.

Examples:

- $\mathcal{G} = \underline{\mathbb{Z}/p\mathbb{Z}}$ constant, $d_m(\mathcal{G}) = 0$, $d_{et}(\mathcal{G}) = 1$;
- $\mathcal{G} = \mu_p$ multiplicative, $d_m(\mathcal{G}) = 1$, $d_{et}(\mathcal{G}) = 0$.

Kisin's conjecture on the set of connected components

$C_\mu(\bar{\rho})^{d_m,d_{et}}$: closed subscheme of $C_\mu(\bar{\rho})$ which paramatrizes finite flat group schemes of multiplicative rank d_m and étale rank d_{et}.

$$C_\mu(\bar{\rho}) = \coprod_{d_m,d_{et}\in\mathbb{N}^2} C_\mu(\bar{\rho})^{d_m,d_{et}}$$

Conjecture (Kisin)

If $\bar{\rho}$ is indecomposable, then $C_\mu(\bar{\rho})^{d_m,d_{et}}$ is connected for any (d_m, d_{et}). In particular, if $\bar{\rho}$ is irreducible, then $C_\mu(\bar{\rho})$ is connected.

Known cases

$n = 2$

- Kisin: K is totally ramified, μ is of particular form,
- Gee, Imai: any K, μ is of particular form,
- Hellmann: $\bar{\rho}$ irreducible, any K, any μ.

Question: What happens for $n \geq 3$?

Group theoretic notations

- $G = \mathrm{Res}_{k|\mathbb{F}_p}(\mathrm{GL}_n)$
- T: a maximal torus of G. We have $T \simeq \mathrm{Res}_{k|\mathbb{F}_p}(\mathbb{G}_m^n)$.
- $X_*(T)$: group of cocharacters of T. $X_*(T) \simeq \mathbb{Z}^{n,\mathrm{Hom}(k,\bar{\mathbb{F}}_p)}$.
- $X_*(T)^+ \subset X_*(T)$: subset of dominant cocharacters of T with respect to a fixed Borel subgroup containing T.

$$X_*(T)^+ \simeq \prod_{\tau \in \mathrm{Hom}(k,\bar{\mathbb{F}}_p)} \mathbb{Z}_+^n$$

$$\mu \mapsto (\mu_\tau)_\tau$$

with $\mu_\tau = (\mu_{\tau,1}, \cdots, \mu_{\tau,n}) \in \mathbb{Z}_+^n$ where
$\mathbb{Z}_+^n := \{(a_i)_i \in \mathbb{Z}^n | a_1 \geq \cdots \geq a_n\}$.

Cartan decomposition inside the affine Grassmannian

Let $L := \bar{\mathbb{F}}_p((u))$, $\mathcal{O}_L := \bar{\mathbb{F}}_p[[u]]$.
We have the Cartan decomposition

$$G(L) = \coprod_{\mu \in X_*(T)^+} G(\mathcal{O}_L) u^{\mu} G(\mathcal{O}_L)$$

This gives a stratification inside the affine Grassmannian:

$$G(L)/G(\mathcal{O}_L) = \coprod_{\mu \in X_*(T)^+} G(\mathcal{O}_L) u^{\mu} G(\mathcal{O}_L)/G(\mathcal{O}_L)$$

satisfying

$$\overline{G(\mathcal{O}_L) u^{\mu} G(\mathcal{O}_L)/G(\mathcal{O}_L)} = \coprod_{\substack{\nu \in X_*(T)^+ \\ \nu \leq \mu}} G(\mathcal{O}_L) u^{\nu} G(\mathcal{O}_L)/G(\mathcal{O}_L).$$

Group theorectic description of Kisin varieties

Recall $N_{\bar\rho} \otimes \bar{\mathbb{F}}_p = (N_{\bar\rho} \otimes_{\mathbb{F}} \bar{\mathbb{F}}_p, \Phi_{\bar\rho})$ the étale φ-module with $\bar{\mathbb{F}}_p$-action associated to $\bar\rho(-1) \otimes \bar{\mathbb{F}}_p$, where

- $N_{\bar\rho} \otimes \bar{\mathbb{F}}_p$ is a free $\bar{\mathbb{F}}_p \otimes_{\mathbb{F}_p} k((u))$-module of rank n;
- $\Phi_{\bar\rho} : N_{\bar\rho} \otimes \bar{\mathbb{F}}_p \to N_{\bar\rho} \otimes \bar{\mathbb{F}}_p$ is semi-linear with respect to φ such that $\varphi^* N_{\bar\rho} \otimes \bar{\mathbb{F}}_p \to N_{\bar\rho} \otimes \bar{\mathbb{F}}_p$ is an isomorphism.

Fix a basis of $N_{\bar\rho} \otimes \bar{\mathbb{F}}_p$, we have

$$(N_{\bar\rho} \otimes \bar{\mathbb{F}}_p, \Phi_{\bar\rho}) \simeq (\bar{\mathbb{F}}_p \otimes_{\mathbb{F}_p} k((u))^n, b\varphi)$$

where $b \in G(L)$ with $G = \mathrm{Res}_{k|\mathbb{F}_p}(\mathrm{GL}_n)$.

The isomorphism class of $(N_{\bar\rho} \otimes \bar{\mathbb{F}}_p, \Phi_{\bar\rho})$ depends on the φ-conjugacy class of $b \in G(L)$.

Group theoretic description of Kisin varieties

For $\mu \in X_*(T)^+$,

$$C_\mu(\bar\rho)(\bar{\mathbb{F}}_p) = \left\{ \begin{array}{c} \mathfrak{M} \subset N_{\bar\rho} \otimes_{\mathbb{F}} \bar{\mathbb{F}}_p \text{ lattice such that the relative} \\ \text{position of } \Phi_{\bar\rho}(\varphi^*\mathfrak{M}) \text{ and } \mathfrak{M} \text{ is controlled by } \mu \end{array} \right\}$$
$$= \{gG(\mathcal{O}_L) \in G(L)/G(\mathcal{O}_L) | g^{-1} b\varphi(g) \in \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)}\}$$
$$:= C_\mu(b)(\bar{\mathbb{F}}_p).$$

This resemble affine Deligne-Lusztig varieties with a different Frobenius φ.

Hypothesis: $\bar\rho$ is absolutely irreducible.
Reason: We can get a good representative of b in its φ-conjugacy class in $G(L)$ by Caruso's classification of simple étale φ-modules over L.

Main results

Theorem (C.-Nie)

Suppose $\bar{\rho}$ is absolutely irreducible, then $C_\mu(\bar{\rho})$ is geometrically connected if one of the following two conditions are satisfied:

1. *K is totally ramified and $n = 3$ (i.e. $G = \mathrm{GL}_3$);*
2. *$\mu = (\mu_\tau)_\tau$ with $\mu_{\tau,2} = \mu_{\tau,3} = \cdots = \mu_{\tau,n}$ for all $\tau \in \mathrm{Hom}(k, \bar{\mathbb{F}}_p)$.*

The second part of the theorem is proved by Hellmann when $n = 2$.

Counter-examples to Kisin's conjecture

We have examples when $\bar{\rho}$ is absolutely irreducible, $C_\mu(\bar{\rho})$ might have two points in the following cases:

- K is totally ramified with $[K:\mathbb{Q}_p] \geq 2p-1$ and $n=4$. Let $G=\mathrm{GL}_4$, $b=u^{(2,0,2,0)}(1243) \in G(\mathbb{F}_p((u)))$ and $\mu=(2p-1,p,p,1)$. Then
$$C_\mu(b)(\mathbb{F}_p) = C_\mu(b)(\bar{\mathbb{F}}_p) = \{u^{(2,1,1,0)}, u^{(1,1,1,1)}\}.$$

- $f(K|\mathbb{Q}_p)=2$, $e(K|\mathbb{Q}_p) \geq p+1$ and $n=3$. Let $G=\mathrm{Res}_{k|\mathbb{F}_p}\mathrm{GL}_3$ with $[k:\mathbb{F}_p]=2$. Choose $\mathbb{F}$ containing k. $G_{|\mathbb{F}} \simeq \mathrm{GL}_3 \times \mathrm{GL}_3$. Let $b=(u^{(2,0,1)}(123), u^{(0,0,1)}) \in G(\mathbb{F}((u)))$ and $\mu=((p+1,0,0),(p,p,0))$, then
$$C_\mu(b)(\mathbb{F}) = C_\mu(b)(\bar{\mathbb{F}}_p) = \{u^\chi, u^{\chi'}\},$$
where $\chi=((1,0,1),(0,0,1))$ and $\chi'=((1,1,0),(1,0,0))$.

Application to the connectedness of deformation rings

Corollary

Suppose $\bar{\rho}$ is absolutely irreducible. For any minuscule $\nu \in X_*(\mathrm{Res}_{K|\mathbb{Q}_p}\mathbb{G}_m^n)^+ \simeq (\mathbb{Z}_+^n)^{\mathrm{Hom}(K,\bar{\mathbb{Q}}_p)}$, the scheme $\mathrm{Spec}(R_{\bar{\rho}}^{\mathrm{fl},\nu}[\frac{1}{p}])$ is connected if one of the following two conditions holds:

1. *K is totally ramified and $n = 3$;*
2. *$\nu_\tau = (1, 0, \cdots, 0)$ or central for all $\tau \in \mathrm{Hom}(K, \bar{\mathbb{Q}}_p)$.*

Strategy of Proof: Semi-module stratification

IAK decomposition inside the affine Grassmannian:

$$G(L)/G(\mathcal{O}_L) = \coprod_{\lambda \in X_*(T)} Iu^{\lambda} G(\mathcal{O}_L)/G(\mathcal{O}_L),$$

where I is the standard Iwahori subgroup of G (i.e., I is the preimage of $B^{op}(\bar{\mathbb{F}}_p)$ under natural map $G(\mathcal{O}_L) \stackrel{u \mapsto 0}{\to} G(\bar{\mathbb{F}}_p)$).

This induces the semi-module decomposition

$$C_{\mu}(b)(\bar{\mathbb{F}}_p) = \sqcup_{\lambda \in X_*(T)} C_{\mu}^{\lambda}(b)(\bar{\mathbb{F}}_p),$$

where each piece $C_{\mu}^{\lambda}(b)$ is locally closed subscheme of $C_{\mu}(b) \times_{\mathbb{F}} \bar{\mathbb{F}}_p$ with $\bar{\mathbb{F}}_p$-points

$$C_{\mu}^{\lambda}(b)(\bar{\mathbb{F}}_p) = (Iu^{\lambda} G(\mathcal{O}_L)/G(\mathcal{O}_L)) \cap C_{\mu}(b)(\bar{\mathbb{F}}_p).$$

Step 1: semi-module strata are connected

Key Proposition

Let $b = u^\eta w$ with $\eta \in X_*(T)$ and $w \in W_0$ such that $bIb^{-1} \subset I$. The following conditions are equivalent:

1. $C_\mu^\lambda(b)$ *is non-empty;*
2. $u^\lambda \in C_\mu(b)(\bar{\mathbb{F}}_p)$;
3. $-\lambda + \eta + w\varphi(\lambda) \leq \mu$.

Under these equivalent conditions, we have $C_\mu^\lambda(b)$ is connected. Moreover, if μ is minuscule, there is a dimension formula for $C_\mu^\lambda(b)$.

Step 2: Connecting different semi-module strata

- K is totally ramified and $n = 3$:
  Construct explicit lines to connect the representatives in different semi-module strata.
- μ is of a particular form:
  Reduce to multi-copy case: $C^{G_\bullet}_{\mu_\bullet}(b_\bullet) \twoheadrightarrow C^G_\mu(b)$ with $\mu_\bullet$ minuscule. The scheme $C^{G_\bullet}_{\mu_\bullet}(b_\bullet)$ has only one 0-dimensional semi-module stratum.

Thank you!