On certain special values of L-functions associated to elliptic curves and real quadratic fields

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And more generally, for any Dirichlet character χ:

$$L(s, E/\mathbf{Q}, \chi) = L(s, f, \chi)$$

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By old results of Shimura, we have:

$$L^{alg}(1, E/\mathbf{Q}, \chi) \in \mathbf{Q}(\chi) \subset \overline{\mathbf{Q}}$$

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For the rest of the lecture, we assume, concerning the data $E/\mathbf{Q}$ and $M/\mathbf{Q}$, the following:

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So in particular, all primes dividing N are unramified in M.

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An important point: the signs of the functional equation for $L(s, E/\mathbf{Q}) = L(s, f)$ and $L(s, E/\mathbf{Q}, \psi) = L(s, f, \psi)$ differs by multiplication by the $\psi(-N)$:

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$$\psi(-N) = \psi(-1) \cdot \psi(N_+) \cdot \psi(N_-)$$

which is -1 by the modified Heegner hypothesis.

Darmon's program

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Darmon's program: to develop an analogue of the theory of Heegner points and Gross-Zagier formulas, in the context of real quadratic extensions of $\mathbf{Q}$; p-adic analytic methods are crucial in Darmon's program, for example in his construction of Stark-Heegner points on elliptic curves.

Preparation for the statement of Main Theorem

To state our main theorem, we first consider a class $\mathcal{C}$ of quadratic Hecke characters $\delta = \otimes_v' \delta_v$ of $\mathbf{A}_M^\times / M^\times$,

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- $\delta_{\mathfrak{l}}$ is nontrivial, i.e. $\delta_{\mathfrak{l}}(\pi_{\mathfrak{l}}) = -1$, for $\mathfrak{l}$ dividing N_-.

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For any such $\delta \in \mathcal{C}$, the sign of the functional equation for $L(s, E/M, \delta) = L(s, \mathbf{f}, \delta)$ is opposite to that of $L(s, E/M) = L(s, \mathbf{f})$.

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Thus the sign of the functional equation for $L(s, E/M, \delta) = L(s, \mathbf{f}, \delta)$ is $+1$.

By the theorem of Friedberg-Hoffstein, there exists infinitely many such quadratic Hecke characters $\delta \in \mathcal{C}$ of $\mathbf{A}_M^\times / M^\times$, satisfying the **nonvanishing condition** $L(1, E/M, \delta) = L(1, \mathbf{f}, \delta) \neq 0$.

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We now define:

$$L^{alg}(1, E/M, \delta) := \frac{D_M^{1/2}(\mathcal{N}_{M/\mathbf{Q}}\mathfrak{c}_\delta)^{1/2} L(1, E/M, \delta)}{(\Omega^+_{E/\mathbf{Q}})^2}$$

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We are interested in studying, for $\delta \in \mathcal{C}$, the numbers $L^{alg}(1, E/M, \delta)$, up to multiplication by squares of (non-zero) rational numbers.

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Suppose that $L'(1, E/M) \neq 0$. Then for any $\delta \in \mathcal{C}$, we have:

$$L^{alg}(1, E/M, \delta) = 2 \times \text{ square of a rational number}$$

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- Our main theorem is consistent with the rank zero case of the Birch and Swinnerton-Dyer conjecture. In particular we expect that the statement of the main theorem should remain valid (at least up to a factor of two), even without the condition that $L'(1, E/M) \neq 0$.
- The original motivation for establishing our main theorem is to understand a certain p-adic Gross-Zagier type formula of Bertolini-Darmon.

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- The condition $L'(1, E/M) \neq 0$ is needed, because Kolyvagin's theorem is used at one and crucial point of the argument (to cancel the transcendental factors coming from the Neron-Tate heights of Heegner points).
- Use results of Ribet-Takahashi concerning degree of modular parametrization of elliptic curve over $\mathbf{Q}$ by modular curve (and similar results in the setting of totally real fields).

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We have the Mazur-Tate-Teitelbaum p-adic L-function $L_p(s, E/\mathbf{Q}) = L_p(s, f)$. From the condition that $E/\mathbf{Q}$ has split multiplicative reduction at p, we have the **exceptional sign change** phenomenon: the sign of the functional equation for $L_p(s, E/\mathbf{Q})$ is opposite to that of $L(s, E/\mathbf{Q})$.

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In general the p-adic interpolation property of p-adic L-function gives:

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Thus if we assume that the sign of the functional equation for $L(s, E/\mathbf{Q})$ being equal to -1, then the sign of the functional equation for $L_p(s, E/\mathbf{Q})$ is equal to $+1$, and we have $L_p(1, E/\mathbf{Q}) = 0, L'_p(1, E/\mathbf{Q}) = 0$,

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Bertolini-Darmon: instead of considering derivative with respect to the s-variable (the cyclotomic variable), consider derivative with respect to the weight variable k, in the context of a Hida family containing f, **and** also in the context of quadratic base change with respect to the real quadratic field M.

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Note that, with the sign of the functional equation for $L(s, E/\mathbf{Q})$ being equal to -1 (and thus the sign of the functional equation for $L(s, E/\mathbf{Q}, \psi)$ is equal to $+1$), we have
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Thus let $\mathcal{F} = \{f_k\}$ be a Hida family containing f.

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Bertolini-Darmon: instead of considering derivative with respect to the s-variable (the cyclotomic variable), consider derivative with respect to the weight variable k, in the context of a Hida family containing f, **and** also in the context of quadratic base change with respect to the real quadratic field M.

Note that, with the sign of the functional equation for $L(s, E/\mathbf{Q})$ being equal to -1 (and thus the sign of the functional equation for $L(s, E/\mathbf{Q}, \psi)$ is equal to $+1$), we have
$L'(1, E/M) = L'(1, E/\mathbf{Q}) \cdot L(1, E/\mathbf{Q}, \psi)$.

Thus let $\mathcal{F} = \{f_k\}$ be a Hida family containing f. Here $f_2 = f$, and for $k \geq 2$, $k \equiv 2 \bmod p-1$ (and k sufficiently close to 2 p-adically), we have that f_k is a Hecke eigenform of weight k.

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Let $\mathfrak{F}$ be the quadratic base change of $\mathcal{F}$ from $GL_{2/\mathbf{Q}}$ to $GL_{2/M}$.

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We have the p-adic L-function $L_p(s, \mathbf{f}_k) = L_p(s, f_k) \cdot L_p(s, f_k, \psi)$ attached to $\mathbf{f}_k$.

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In addition we have the two variable p-adic L-function $L_p(s, \mathfrak{F}) = L_p(s, \{\mathbf{f}_k\})$ associated to the parallel weights Hida family $\mathfrak{F}$, that interpolates the p-adic L-function $L_p(s, \mathbf{f}_k)$ associated to each $\mathbf{f}_k$.

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In addition we have the two variable p-adic L-function $L_p(s, \mathfrak{F}) = L_p(s, \{\mathbf{f}_k\})$ associated to the parallel weights Hida family $\mathfrak{F}$, that interpolates the p-adic L-function $L_p(s, \mathbf{f}_k)$ associated to each $\mathbf{f}_k$. We denote this simply as $L_p(s, k)$. We are interested in the values of this two-variable p-adic L-function on the line $s = k/2$.

Application to a p-adic Gross-Zagier type formula of Bertolini-Darmon

We have the following theorem: as before assume that $E/\mathbf{Q}$ has split-multiplicative reduction at p, and that the sign of the functional equation for $L(s, E/\mathbf{Q})$ is equal to -1.

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$$\frac{d^2}{dk^2} L_p(k/2, k)\Big|_{k=2} = 2 \cdot (\log_{E,p}(\mathbf{P}))^2$$

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Here $\log_{E,p}$ is the p-adic logarithm on $E/\mathbf{Q}_p$ defined using Tate's p-adic uniformization of $E/\mathbf{Q}_p$, and $\mathbf{P} \in E(M) \otimes \mathbf{Q}$; $\mathbf{P}$ is non-torsion iff $L'(1, E/M) \neq 0$ (thus if $L'(1, E/M) = 0$ then both sides of the formula are zero).

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This formula was proved by Bertolini-Darmon in their paper *Hida families and rational points on elliptic curves*, in the case where $E/\mathbf{Q}$ satisfies the extra condition that it has multiplicative reduction at some prime $q|N_+$.

Without this extra condition, I had shown in a previous work that the following formula holds:

Application to a p-adic Gross-Zagier type formula of Bertolini-Darmon

$$\frac{d^2}{dk^2} L_p(k/2, k)\Big|_{k=2} = \ell \cdot (\log_{E,p}(\mathbf{P}))^2$$

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where $\mathbf{P} \in E(M) \otimes \mathbf{Q}$ as before, and ℓ is a rational number, with:

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where $\mathbf{P} \in E(M) \otimes \mathbf{Q}$ as before, and ℓ is a rational number, with:

$$\ell \equiv L^{alg}(1, E/M, \delta) \text{ mod } (\mathbf{Q}^\times)^2$$

for any $\delta \in \mathcal{C}$ with $L(1, E/M, \delta) \neq 0$.

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for any $\delta \in \mathcal{C}$ with $L(1, E/M, \delta) \neq 0$.

Our main theorem on special values thus amounts to saying that ℓ is two times the square of a rational number. Consequently the Bertolini-Darmon formula holds without the extra condition.