On certain special values of $L$-functions associated to elliptic curves and real quadratic fields

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Let $E/\mathbb{Q}$ be an elliptic curve over $\mathbb{Q}$, $N = \text{cond}(E/\mathbb{Q})$. 
Introduction

Let \( E/Q \) be an elliptic curve over \( \mathbb{Q} \), \( N = \text{cond}(E/Q) \).

By the modularity theorem, we have the weight two Hecke eigenform \( f = f_E \) of level \( N \) that is associated to \( E/Q \).
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In terms of \( L \)-functions:

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L(s, E/Q) = L(s, f)
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By the modularity theorem, we have the weight two Hecke eigenform $f = f_E$ of level $N$ that is associated to $E/Q$.

In terms of $L$-functions:

$$L(s, E/Q) = L(s, f)$$

And more generally, for any Dirichlet character $\chi$:

$$L(s, E/Q, \chi) = L(s, f, \chi)$$
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We assume that $\chi$ is even. Define:

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where $c_\chi$ is the conductor of the Dirichlet character $\chi$, $\tau(\chi)$ is the Gauss sum of $\chi$, and

$$\Omega^+_E/Q = \int_E(R) |\omega_E/Q|$$

for a choice of global invariant 1-form $\omega_E/Q$ of $E/Q$. By old results of Shimura, we have:

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By old results of Shimura, we have:

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In particular, if $\chi$ is an even quadratic Dirichlet character, then we have:

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For the rest of the lecture, we assume, concerning the data $E/\mathbb{Q}$ and $M/\mathbb{Q}$, the following:
Modified Heegner Hypothesis

\[ N = \text{cond}(E/Q) \] can be factorized as \( N = N_+ \cdot N_- \), where

- \( N_- \) is square-free, and is equal to a product of an odd number of distinct primes.
- All primes dividing \( N_+ \) split in \( M \), while all primes dividing \( N_- \) are inert in \( M \).
- So in particular, all primes dividing \( N \) are unramified in \( M \).
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Consider $E/M$. It is again modular, by the theory of quadratic base change on the automorphic side.
Quadratic Base Change

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$$L(s, E/M) = L(s, E/Q) \cdot L(s, E/Q, \psi)$$

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At the level of $L$-functions, we have, with $\psi$ being the even quadratic Dirichlet character that corresponds to $M/\mathbb{Q}$:

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We have the equality of $L$-functions:

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The conductor of $E/M$ and $f$ is given by $N\mathcal{O}_M$. 

An important point: the signs of the functional equation for $L(s, E/Q) = L(s, f)$ and $L(s, E/Q, \psi) = L(s, f, \psi)$ differs by multiplication by the $\psi(\cdot)$:

$$\psi(-N) = \psi(-1) \cdot \psi(N+1) \cdot \psi(N-1)$$

which is $-1$ by the modified Heegner hypothesis.
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Darmon’s program

Since $L(s, E/M) = L(s, E/Q) \cdot L(s, E/Q, \psi)$, it follows that the sign of the functional equation for $L(s, E/M)$ is always equal to $-1$. 

In particular $L(1, E/M) = 0$.

Arithmetic significance of $L'(1, E/M)$?

Darmon’s program: to develop an analogue of the theory of Heegner points and Gross-Zagier formulas, in the context of real quadratic extensions of $\mathbb{Q}$; $p$-adic analytic methods are crucial in Darmon’s program, for example in his construction of Stark-Heegner points on elliptic curves.
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- $\delta_v$ is trivial for $v|\infty$.
- $\delta_l$ is trivial for $l$ dividing $N_+$.
- $\delta_l$ is nontrivial, i.e. $\delta_l(\pi_l) = -1$, for $l$ dividing $N_-$.
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For any such $\delta \in \mathcal{C}$, the sign of the functional equation for $L(s, E/M, \delta) = L(s, f, \delta)$ is opposite to that of $L(s, E/M) = L(s, f)$. 

By the theorem of Friedberg-Hoffstein, there exists infinitely many such quadratic Hecke characters $\delta \in \mathcal{C}$ of $A \times M / M$, satisfying the nonvanishing condition $L(1, E/M, \delta) = L(1, f, \delta) \neq 0$. 
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Thus the sign of the functional equation for $L(s, E/M, \delta) = L(s, f, \delta)$ is $+1$. 

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We now define:

\[ L_{\text{alg}}(1, E/M, \delta) := \frac{D_{M}^{1/2}(\mathcal{N}_{M/Q}c_{\delta})^{1/2}L(1, E/M, \delta)}{(\Omega_{E/Q}^{+})^{2}} \]

Techniques of Shimura allow one to show that \( L_{\text{alg}}(1, E/M, \delta) \in \mathbb{Q}. \)
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We are interested in studying, for \( \delta \in \mathcal{C} \), the numbers \( L_{\text{alg}}(1, E/M, \delta) \), up to multiplication by squares of (non-zero) rational numbers.
Our main theorem is as follows (to appear in the Transactions of the AMS):

Suppose that $L'(1, E/M) \neq 0$. Then for any $\delta \in C$, we have:

$L_{\text{alg}}(1, E/M, \delta) = 2 \times \text{square of a rational number}$
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- For any $\delta \in \mathcal{C}$, we have that $\delta|_{A^\times}^\times$ is nontrivial;
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• For any $\delta \in C$, we have that $\delta|_{A_Q}$ is nontrivial; thus Waldspurger’s central $L$-value formula could not be directly applied to the $L$-value $L(1, E/M, \delta)$.

• Our main theorem is consistent with the rank zero case of the Birch and Swinnerton-Dyer conjecture. In particular we expect that the statement of the main theorem should remain valid (at least up to a factor of two), even without the condition that $L'(1, E/M) \neq 0$.

• The original motivation for establishing our main theorem is to understand a certain $p$-adic Gross-Zagier type formula of Bertolini-Darmon.
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Some ideas on the proof:

• Use the Friedberg-Hoffstein theorem to construct suitable imaginary quadratic extensions of $\mathbb{Q}$ and $\mathbb{CM}$-extensions of $\mathbb{M}$, where Gross-Zagier formulas (as generalized by Shouwu Zhang) for central $L$-values and central $L$-derivatives are applicable.

• Then express $L(1, E/M, \delta)$ in terms of these auxiliary central $L$-values and central $L$-derivatives.

• The condition $L'(1, E/M) \neq 0$ is needed, because Kolyvagin's theorem is used at one and crucial point of the argument (to cancel the transcendental factors coming from the Neron-Tate heights of Heegner points).

• Use results of Ribet-Takahashi concerning degree of modular parametrization of elliptic curve over $\mathbb{Q}$ by modular curve (and similar results in the setting of totally real fields).
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For the rest of the talk, assume $N_-$ is equal to a single odd prime $p$ (the modified Heegner hypothesis is still in force).
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and hence we always have $L_p(1, E/\mathbb{Q}) = 0$ irregardless of the value of $L(1, E/\mathbb{Q})$, i.e. a trivial zero.
Application to a $p$-adic Gross-Zagier type formula of Bertolini-Darmon

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Thus if we assume that the sign of the functional equation for $L(s, E/Q)$ being equal to $-1$, then the sign of the functional equation for $L_p(s, E/Q)$ is equal to $+1$, and we have $L_p(1, E/Q) = 0$, $L'_p(1, E/Q) = 0$, so it is of interest to study the second derivative $L''_p(1, E/Q)$. 
Application to a $p$-adic Gross-Zagier type formula of Bertolini-Darmon

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Bertolini-Darmon: instead of considering derivative with respect to the $s$-variable (the cyclotomic variable), consider derivative with respect to the weight variable $k$, in the context of a Hida family containing $f$, 

Thus let $F = \{f_k\}$ be a Hida family containing $f$, and for $k \geq 2$, $k \equiv 2 \pmod{p-1}$ (and $k$ sufficiently close to $2p$-adically), we have that $f_k$ is a Hecke eigenform of weight $k$. 

Application to a $p$-adic Gross-Zagier type formula of Bertolini-Darmon

Bertolini-Darmon: instead of considering derivative with respect to the $s$-variable (the cyclotomic variable), consider derivative with respect to the weight variable $k$, in the context of a Hida family containing $f$, and also in the context of quadratic base change with respect to the real quadratic field $M$. 
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Note that, with the sign of the functional equation for $L(s, E/Q)$ being equal to $-1$ (and thus the sign of the functional equation for $L(s, E/Q, \psi)$ is equal to $+1$), we have $L'(1, E/M) = L'(1, E/Q) \cdot L(1, E/Q, \psi)$. 
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Thus let $\mathcal{F} = \{f_k\}$ be a Hida family containing $f$. Here $f_2 = f$, and for $k \geq 2$, $k \equiv 2 \mod p - 1$ (and $k$ sufficiently close to 2 $p$-adically), we have that $f_k$ is a Hecke eigenform of weight $k$. 
Application to a $p$-adic Gross-Zagier type formula of Bertolini-Darmon

Let $\mathcal{F}$ be the quadratic base change of $\mathcal{F}$ from $GL_2/\mathbb{Q}$ to $GL_2/M$. 
Application to a \( p \)-adic Gross-Zagier type formula of Bertolini-Darmon

Let \( \mathcal{F} \) be the quadratic base change of \( \mathcal{F} \) from \( GL_2/\mathbb{Q} \) to \( GL_2/M \). Thus \( \mathcal{F} = \{ f_k \} \) is a Hida family of parallel weights Hilbert modular Hecke eigenforms over \( M \);
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Let $\mathcal{F}$ be the quadratic base change of $\mathcal{F}$ from $GL_2/\mathbb{Q}$ to $GL_2/M$. Thus $\mathcal{F} = \{f_k\}$ is a Hida family of parallel weights Hilbert modular Hecke eigenforms over $M$; one has that $f_k$ is the base change of $f_k$ from $GL_2/\mathbb{Q}$ to $GL_2/M$; in particular $f_2 = f$. 
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We have the $p$-adic $L$-function $L_p(s, f_k) = L_p(s, f_k) \cdot L_p(s, f_k, \psi)$ attached to $f_k$. 
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Let $\mathcal{F}$ be the quadratic base change of $\mathcal{F}$ from $GL_2/\mathbb{Q}$ to $GL_2/M$. Thus $\mathcal{F} = \{f_k\}$ is a Hida family of parallel weights Hilbert modular Hecke eigenforms over $M$; one has that $f_k$ is the base change of $f_k$ from $GL_2/\mathbb{Q}$ to $GL_2/M$; in particular $f_2 = f$.

We have the $p$-adic $L$-function $L_p(s, f_k) = L_p(s, f_k) \cdot L_p(s, f_k, \psi)$ attached to $f_k$.

In addition we have the two variable $p$-adic $L$-function $L_p(s, \mathcal{F}) = L_p(s, \{f_k\})$ associated to the parallel weights Hida family $\mathcal{F}$, that interpolates the $p$-adic $L$-function $L_p(s, f_k)$ associated to each $f_k$. 
Application to a $p$-adic Gross-Zagier type formula of Bertolini-Darmon

Let $\mathcal{F}$ be the quadratic base change of $F$ from $GL_2/\mathbb{Q}$ to $GL_2/M$. Thus $\mathcal{F} = \{f_k\}$ is a Hida family of parallel weights Hilbert modular Hecke eigenforms over $M$; one has that $f_k$ is the base change of $f_k$ from $GL_2/\mathbb{Q}$ to $GL_2/M$; in particular $f_2 = f$.

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Let $\mathcal{F}$ be the quadratic base change of $\mathcal{F}$ from $GL_2/\mathbb{Q}$ to $GL_2/M$. Thus $\mathcal{F} = \{f_k\}$ is a Hida family of parallel weights Hilbert modular Hecke eigenforms over $M$; one has that $f_k$ is the base change of $f_k$ from $GL_2/\mathbb{Q}$ to $GL_2/M$; in particular $f_2 = f$.

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In addition we have the two variable $p$-adic $L$-function $L_p(s, \mathcal{F}) = L_p(s, \{f_k\})$ associated to the parallel weights Hida family $\mathcal{F}$, that interpolates the $p$-adic $L$-function $L_p(s, f_k)$ associated to each $f_k$. We denote this simply as $L_p(s, k)$. We are interested in the values of this two-variable $p$-adic $L$-function on the line $s = k/2$. 
Application to a $p$-adic Gross-Zagier type formula of Bertolini-Darmon

We have the following theorem: as before assume that $E/\mathbb{Q}$ has split-multiplicative reduction at $p$, and that the sign of the functional equation for $L(s, E/\mathbb{Q})$ is equal to $-1$. 
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We have the following theorem: as before assume that \( E/\mathbb{Q} \) has split-multiplicative reduction at \( p \), and that the sign of the functional equation for \( L(s, E/\mathbb{Q}) \) is equal to \(-1\). Then we have the formula:

\[
\frac{d^2}{dk^2} L_p(k/2, k) \bigg|_{k=2} = 2 \cdot (\log_{E,p}(P))^2
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Application to a $p$-adic Gross-Zagier type formula of Bertolini-Darmon

We have the following theorem: as before assume that $E/Q$ has split-multiplicative reduction at $p$, and that the sign of the functional equation for $L(s, E/Q)$ is equal to $-1$. Then we have the formula:

$$\frac{d^2}{dk^2} L_p(k/2, k) \bigg|_{k=2} = 2 \cdot (\log_{E,p}(P))^2$$

Here $\log_{E,p}$ is the $p$-adic logarithm on $E/Q_p$ defined using Tate’s $p$-adic uniformization of $E/Q_p$, and $P \in E(M) \otimes Q$;
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Here $\log_{E,p}$ is the $p$-adic logarithm on $E/\mathbb{Q}_p$ defined using Tate’s $p$-adic uniformization of $E/\mathbb{Q}_p$, and $P \in E(M) \otimes \mathbb{Q}$; $P$ is non-torsion iff $L'(1, E/M) \neq 0$ (thus if $L'(1, E/M) = 0$ then both sides of the formula are zero).
Application to a \( p \)-adic Gross-Zagier type formula of Bertolini-Darmon

This formula was proved by Bertolini-Darmon in their paper *Hida families and rational points on elliptic curves*,

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This formula was proved by Bertolini-Darmon in their paper *Hida families and rational points on elliptic curves*, in the case where $E/\mathbb{Q}$ satisfies the extra condition that it has multiplicative reduction at some prime $q|N_+$. 
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This formula was proved by Bertolini-Darmon in their paper *Hida families and rational points on elliptic curves*, in the case where $E/\mathbb{Q}$ satisfies the extra condition that it has multiplicative reduction at some prime $q|N_+$. 

Without this extra condition, I had shown in a previous work that the following formula holds:
Application to a $p$-adic Gross-Zagier type formula of Bertolini-Darmon

$$\left. \frac{d^2}{dk^2} L_p(k/2, k) \right|_{k=2} = \ell \cdot (\log_{E,p}(P))^2$$
Application to a $p$-adic Gross-Zagier type formula of Bertolini-Darmon

$$\frac{d^2}{dk^2} L_p(k/2, k) \bigg|_{k=2} = \ell \cdot (\log_{E,p}(P))^2$$

where $P \in E(M) \otimes \mathbb{Q}$ as before, and $\ell$ is a rational number, with:
Application to a $p$-adic Gross-Zagier type formula of Bertolini-Darmon

$$\frac{d^2}{dk^2} L_p(k/2, k) \bigg|_{k=2} = \ell \cdot (\log_{E,p}(P))^2$$

where $P \in E(M) \otimes \mathbb{Q}$ as before, and $\ell$ is a rational number, with:

$$\ell \equiv L^{alg}(1, E/M, \delta) \mod (\mathbb{Q}^\times)^2$$

for any $\delta \in \mathcal{C}$ with $L(1, E/M, \delta) \neq 0$. 
Application to a $p$-adic Gross-Zagier type formula of Bertolini-Darmon

$$\left. \frac{d^2}{dk^2} L_p(k/2, k) \right|_{k=2} = \ell \cdot (\log_{E,p}(P))^2$$

where $P \in E(M) \otimes \mathbb{Q}$ as before, and $\ell$ is a rational number, with:

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for any $\delta \in \mathcal{C}$ with $L(1, E/M, \delta) \neq 0$.

Our main theorem on special values thus amounts to saying that $\ell$ is two times the square of a rational number. Consequently the Bertolini-Darmon formula holds without the extra condition.