A Serre weight conjecture for mod p Hilbert modular forms

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Before I begin

A lot of what I say today is joint work with F. Diamond (DS).

Introduction

Let p be a rational prime. Let

$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

be a continuous, odd and irreducible representation over $\overline{\mathbb{F}}_p$.

Serre's conjecture

J.-P. Serre (1987) defined/specified

- $k(\overline{\rho}) \geq 2$
- $N(\overline{\rho}) \geq 1$, the Artin conductor prime to p,
- (a character $(\mathbb{Z}/N(\overline{\rho})\mathbb{Z})^\times \to \overline{\mathbb{F}}_p^\times$)

and conjectured that there should be a cuspidal modular eigenform f of weight $k(\overline{\rho})$ and level $N(\overline{\rho})$ such that (for a choice of $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$),

$$f \rightsquigarrow \overline{\rho}_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\rho_f} \mathrm{GL}_2(\overline{\mathbb{Z}}_p) \twoheadrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

is isomorphic to $\overline{\rho}$.

Serre's conjecture in the Hilbert case

Generalising the weight part of Serre's conjecture ("if $\overline{\rho}$ is modular, of what weight exactly?") to the setting of mod p HMFs (of regular weights) was initiated by Buzzard-Diamond-Jarvis (2010).

The BDJ conjecture ($p > 2$) has been proved almost completely by Gee, Liu and Savitt.

Today, I will formulate a Serre weight conjecture for all weights within geometric theory of Hilbert modular forms (Katz, Goren, Andreatta-Goren,...).

We are generalising Edixhoven's reformulation (1992) of Serre's conjecture.

To start with

Fix $\overline{\mathbb{Q}}$, $\overline{\mathbb{Q}}_p$ and $\overline{\mathbb{F}}_p$ and fix an embedding

$$\imath : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$$

once for all.

Let

$$\Sigma = \imath \circ \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}) = \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}_p).$$

Suppose that there is only one prime v in $\mathcal{O}_F$ above p. Fix a uniformiser at v.

$\sigma_\tau(j)$

If $F_v^0 \subset F_v$ denote the maximal unramified extension of $\mathbb{Q}_p$ of degree f_v, then the restriction to F_v^0 defines a surjection

$$\Sigma = \mathrm{Hom}_{\mathbb{Q}_p}(F_v, \overline{\mathbb{Q}}_p) = \Sigma \twoheadrightarrow \Sigma^0 = \mathrm{Hom}_{\mathbb{Q}_p}(F_v^0, \overline{\mathbb{Q}}_p) \circlearrowleft \phi.$$

Suppose that $\{\sigma_\tau(1), \ldots, \sigma_\tau(e_v)\} \subset \Sigma$ is the pre-image of τ in Σ^0; they are the (ordered) roots of the Eisenstein polynomial $\tau \circ E$.

To sum up, every embedding σ in Σ is determined uniquely by a pair of τ in Σ^0 and $1 \leq j \leq e_v$.

An index shifting operator Φ

Define an index shift Φ on Σ:

$$\underbrace{\cdots \rightsquigarrow \sigma_{\phi^{-1}\circ\tau}(e_v)}_{\phi^{-1}\circ\tau} \overset{\phi}{\rightsquigarrow} \underbrace{\sigma_\tau(1) \rightsquigarrow \cdots \rightsquigarrow \sigma_\tau(e_v)}_{\tau} \overset{\phi}{\rightsquigarrow} \underbrace{\sigma_{\phi\circ\tau}(1) \rightsquigarrow \cdots}_{\phi\circ\tau}$$

Models of HMFs

Let

$$G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$$

and let

- $\Gamma \subset G(\mathbb{A}^\infty)$ maximal compact hyperspecial at p, which we always assume sufficiently small,
- (Rapoport/Deligne-Pappas/Pappas-Rapoport) a smooth integral model Y_Γ over $\overline{\mathbb{Z}}_p$ for

$$G(\mathbb{Q}) \backslash (\mathbb{C} - \mathbb{R})^\Sigma \times G(\mathbb{A}^\infty)/\Gamma$$

Pappas-Rapoport

The Pappas-Rapoport integral model Y_Γ parametrises HBAVs $s : A \to S$ of PEL type (in the sense of DP) such that the locally free $\mathscr{O}_S$-module of rank $[F : \mathbb{Q}] = d = e_v f_v$ with $\mathscr{O}_F$ action

$$\omega = s_* \Omega_{A/S} = \bigoplus_{\tau \in \Sigma^0} \omega_\tau$$

where ω_τ (locally free $\mathscr{O}_S$-module of rank e_v) comes equipped with a filtration (a 'complete flag')

$$0 = \omega_\tau(0) \subset \omega_\tau(1) \subset \cdots \subset \omega_\tau(e_v) = \omega_\tau$$

where $\mathscr{O}_F$ acts via the embedding $\sigma_\tau(j)$ on the rank 1 subquotient

$$\omega_{\sigma_\tau(j)} := \omega_\tau(j)/\omega_\tau(j-1).$$

When $e_v = 1$, ω is locally free module of rank 1 over $\mathscr{O}_F \otimes_{\mathbb{Z}} \mathscr{O}_S$.

Automorphic bundle $\mathscr{A}_{(k,\ell)}$

- associated to $(k,\ell) \in \mathbb{Z}^{\Sigma} \times \mathbb{Z}^{\Sigma}$, we have the automorphic line bundle

$$\mathscr{A}_{(k,\ell)} = \bigotimes_{\sigma \in \Sigma} \omega_{\sigma}^{k_\sigma} \otimes \delta_{\sigma}^{\ell_\sigma}$$

where ω_σ is a locally-free-of-rank-1-over-$\mathscr{O}_{Y_\Gamma}$ piece of ω on which $\mathscr{O}_F$ acts via σ, and where

$$\delta = \bigwedge^{2}_{\mathscr{O}_F \otimes_{\mathbb{Z}} \mathscr{O}_{Y_\Gamma}} R^1 s_* \Omega^{\bullet}_{A/Y_\Gamma}$$

is free of rank 1 over $\mathscr{O}_F \otimes_{\mathbb{Z}} \mathscr{O}_{Y_\Gamma}$ and δ_σ is defined similarly.

Mod p HMFs

The space of mod p Hilbert modular forms of weight (k, ℓ) and of level Γ is defined to be

$$H^0(\overline{Y}_\Gamma, \mathscr{A}_{(k,\ell)})$$

where $\overline{Y}_\Gamma = Y_\Gamma \times \overline{\mathbb{F}}_p$.

$F \neq \mathbb{Q}$

When $F \neq \mathbb{Q}$,

- every Hilbert modular form of weight (k, ℓ) over $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$ has its weight paritious, i.e. $k_\sigma + 2\ell_\sigma$ is independent of σ.

  There are a lot more mod p Hilbert modular forms that are not in the image of

$$H^0(Y_\Gamma, \mathscr{A}_{(k,\ell)}) \to H^0(\overline{Y}_\Gamma, \mathscr{A}_{(k,\ell)}).$$

- In stark contract to the case $F = \mathbb{Q}$, there are lots of mod p Hilbert modular forms of '(partially) negative weights'.

Example ($\ell = 0$)

Reduzzi-Xiao's partial Hasse invariants H_σ: the Verschibung $V = (\mathrm{Fr}_{A^\vee})^\vee : A^{(p)} \to A$ gives rise to

$$V^* : \overline{\omega}_\tau \to (\mathrm{Fr}^* \overline{\omega})_\tau = \overline{\omega}^p_{\phi^{-1}\circ\tau}$$

which breaks up into maps of sheaves on $\overline{\omega}_{\sigma_\tau(j)} = \overline{\omega}_\tau(j)/\overline{\omega}_\tau(j-1)$

$$\cdots \to \overline{\omega}_{\sigma_\tau(e_v)} \longrightarrow \overline{\omega}_{\sigma_\tau(e_v-1)} \to \cdots \to \overline{\omega}_{\sigma_\tau(1)} \longrightarrow \overline{\omega}^p_{\sigma_{\phi^{-1}\circ\tau}(e_v)} \to \cdots \qquad \text{(arc } V^*: \overline{\omega}_{\sigma_\tau(e_v)} \to \overline{\omega}^p_{\sigma_{\phi^{-1}\circ\tau}(e_v)}\text{)}$$

A mod p Hilbert modular form H_σ of level Γ and of weight

$$h_\sigma = \begin{cases} 1\Phi^{-1}\sigma + (-1)\sigma & \text{if } \sigma = \sigma_\tau(j) \text{ for } e_v \geq j \geq 2, \\ p\Phi^{-1}\sigma + (-1)\sigma & \text{if } \sigma = \sigma_\tau(1) \end{cases}$$

Automorphic Galois representations

Theorem (DS)

Let f be an element $H^0(\overline{Y}_\Gamma, \mathscr{A}_{(k,\ell)})$ and S be a finite set of finite places in F, containing all v dividing p and all v such that $\mathrm{GL}_2(\mathscr{O}_{F_v}) \not\subset \Gamma$.
Suppose that

$$T_v f = \alpha_v f$$

and

$$S_v f = \beta_v f$$

for all v not in S. Then there exists a continuous representation

$$\overline{\rho}_f : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

which is unramified outside S and the characteristic polynomial in X of $\overline{\rho}_f(\mathrm{Frob}_v)$ is

$$X^2 - \alpha_v X + \beta_v \mathrm{N}_{F/\mathbb{Q}}(v).$$

Remark

The novelty of our theorem is that (k, ℓ) does not have to satisfy the parity condition ($k_\sigma + 2\ell_\sigma$ is independent of σ in Σ). The parity case is known by Emerton-Reduzzi-Xiao ($e_v = 1$), RX ($e_v \geq 1$), and Goldring-Koskivirta (general Shimura varieties).

Idea of our proof for the theorem when $e_v = 1$

Recall that not every $\overline{\rho}$ arises as the reduction of a characteristic zero eigenform of level prime to p.

How do we deal with HMFs of non-paritous weight? We lift mod p HMFs of parallel weight but of level $\Gamma \cap \Gamma_1(p)$.

The idea then is to establish congruences, i.e., find an eigenform of parallel weight $N + 2$ and level $\Gamma \cap \Gamma_1(p)$ which is congruent mod p (and Hasse) to f and which can be lifted to a characteristic zero eigenform when N is sufficiently large (the ampleness of a line bundle over the minimal compactification X_Γ).

When $e_v > 1$, it gets more complicated (but proved in my forthcoming joint work with Diamond).

Conjecture (Folklore)

Let

$$\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

be totally odd, continuous and irreducible. Then $\overline{\rho}$ is modular in the sense above.

In preparation of the DS conjecture

Definition (the Diamond-Kassaei minimal cone)

Let $\Xi^{\mathrm{DK}} \subset \mathbb{Z}^{\Sigma}$ be the set of $k = \sum_{\sigma} k_{\sigma}\sigma$ such that

- $pk_{\sigma} \geq k_{\Phi^{-1}\sigma}$ if σ is of the form $\sigma_{\tau}(1)$,
- $k_{\sigma} \geq k_{\Phi^{-1}\sigma}$ if σ is of the form $\sigma_{\tau}(j)$ for $2 \leq j \leq e_{v}$.

And let

$$\Xi = \{k \in \Xi^{\mathrm{DK}} \mid k_{\sigma} \geq 1 \text{ for every } \sigma \text{ in } \Sigma\}$$

Definition

$$k \geq_{\mathrm{H}} k'$$

if $k - k'$ is a non-negative integer linear combination of the weights h_{σ} of the partial Hasse invariants.

Conjecture

Conjecture (DS)

Let

$$\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p),$$

totally odd, continuous, irreducible.

Fix ℓ in $\mathbb{Z}^{\Sigma}$. Then there exists $k(\overline{\rho}, \ell)$ lying in Ξ satisfying the following conditions:

- $\overline{\rho}$ is modular of weight (k, ℓ) if and only if $k \geq_{\mathrm{H}} k(\overline{\rho}, \ell)$
- if $k \in \Xi$, then $k \geq_{\mathrm{H}} k(\overline{\rho}, \ell)$ if and only if $\overline{\rho}_v = \overline{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_v)}$ has a crystalline lift of weight (k, ℓ), i.e. of Hodge-Tate weight $(k + \ell - 1, \ell)$.

Remark/Companion forms

Assuming $\overline{\rho}$ is (geometrically) modular, the existence of $k(\overline{\rho}, \ell)$ is suggested by DK– the weight filtration $w(f)$ of a mod p HMF f lies in Ξ^{DK}.

The first condition: 'minimality'

Assuming the existence of $k(\overline{\rho}, \ell) \in \Xi$ satisfying the first condition, one sees that

• the conjecture implies the folklore conjecture earlier,

• the forthcoming work of Diamond and Kassaei proves that the conjecture (i.e. the second condition) is equivalent to

Conjecture $*$ (DS)

Suppose that $\overline{\rho}$ is irreducible and modular. If $k \in \Xi$, then $\overline{\rho}$ is modular of weight (k, ℓ) if and only if $\overline{\rho}_v$ has crystalline lift of weight (k, ℓ).

The second condition: p-adic Hodge-theory

The second condition is suggested by the Breuil-Mézard conjecture and modular representation theory of $\mathrm{GL}_2(\mathbb{F}_v)$– it is somehow the underlying theme of DKS.

The qualification '$k \in \Xi$' in the second condition is needed– when $k \notin \Xi$, the condition $k \geq_{\mathrm{H}} k(\overline{\rho}, \ell)$ does not imply that $\overline{\rho}_v$ has crystalline lift of weight (k, ℓ). It is possible to write down an example.

Example when $F = \mathbb{Q}$ and $\ell = 0$

$$\Xi^{\mathrm{DK}} = \{k \geq 0\},$$

$$\Xi = \{k \geq 1\},$$

$$k \geq_{\mathrm{H}} k' \text{ if } k - k' = (p-1)n \geq 0.$$

There exists $k(\overline{\rho}) \geq 1$ such that the following are equivalent:

- $\overline{\rho}$ is modular of weight k,
- $k \geq_{\mathrm{H}} k(\overline{\rho})$,
- $\overline{\rho}_p$ has a crystalline lift of weight $(k, 0)$,

for every $k \geq 1$.

$\rightsquigarrow$ $k(\overline{\rho})$ is the smallest possible weight for which $\overline{\rho}$ is modular, which is Edixhoven's theorem.

How is our conjecture related to the Buzzard-Diamond-Jarvis conjecture?

Algebraic modular = Geometric modular?

Conjecture (DS)

Let $(k, \ell) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$ and $k_\sigma \geq 2$ for every σ in Σ. If $\overline{\rho}$ is algebraic modular of weight (k, ℓ), i.e., of representation weight

$$V_{k,1-k-\ell} = \bigotimes_\sigma \mathrm{Sym}^{k_\sigma - 2} \det{}^{1-k_\sigma-\ell_\sigma}(V_{\mathrm{st}} \otimes_\sigma \overline{\mathbb{F}}_p),$$

(where V_{st} is the standard representation of $\mathrm{GL}_2(\mathbb{F}_v)$ on two copies of $\mathbb{F}_v$) then $\overline{\rho}$ is modular of weight (k, ℓ).

If furthermore $k \in \Xi$, the converse holds. This is false if $k \notin \Xi$!

Remarks

We know that if $\overline{\rho}$ is algebraic modular of paritious weight (k, ℓ), then $\overline{\rho}$ is modular of weight (k, ℓ).

By our construction of modular Galois representations, if $\overline{\rho}$ is modular of some weight, $\overline{\rho}$ is algebraic modular of some weight.

Conjecture (Recap)

Conjecture (DS)

Let

$$\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p),$$

totally odd, continuous, irreducible.

Fix ℓ in $\mathbb{Z}^{\Sigma}$. Then there exists $k(\overline{\rho}, \ell)$ lying in Ξ satisfying the following conditions:

- $\overline{\rho}$ is modular of weight (k, ℓ) if and only if $k \geq_{\mathrm{H}} k(\overline{\rho}, \ell)$
- if $k \in \Xi$, then $k \geq_{\mathrm{H}} k(\overline{\rho}, \ell)$ if and only if $\overline{\rho}_v = \overline{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_v)}$ has a crystalline lift of weight (k, ℓ), i.e. of Hodge-Tate weight $(k + \ell - 1, \ell)$.

(recap)

Assuming 'minimal weight' $k(\overline{\rho}, \ell) \in \Xi$ exists (that satisfies the first condition), the second condition is equivalent to

Conjecture $*$ (DS)

Suppose that $\overline{\rho}$ is irreducible and modular. If $k \in \Xi$, then $\overline{\rho}$ is modular of weight (k, ℓ) if and only if $\overline{\rho}_v$ has crystalline lift of weight (k, ℓ).

Evidence 1: when $[F : \mathbb{Q}] = 2$

Suppose $[F : \mathbb{Q}] = [F_v : \mathbb{Q}_p] = 2$.

- If $e_v = 1$ and $f_v = 2$, then

$$\Sigma = \{\sigma_\tau, \sigma_{\phi\circ\tau} = \sigma_{\phi^{-1}\circ\tau}\}$$

  when τ is fixed.

- If $e_v = 2$ and $f_v = 1$, then

$$\Sigma = \{\sigma(1), \sigma(2)\}$$

Theorem (DS)

Let $2 < r \leq p$. Suppose that $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is irreducible and modular.

If $\overline{\rho}_v$ has crystalline lift of weight

$$(k, \ell) = ((r, 1), (0, 0))$$

(=HT weight $((r-1, 0), (0, 0))$) then $\overline{\rho}$ is (geometric) modular of weight

$$((r, 1), (0, 0)).$$

It is only for simplicity that ℓ is assumed to be $(0, 0)$.

Evidence 2: a mod p analogue of the Fontaine-Mazur conjecture

Theorem (S)

Suppose that

$$\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

is continuous and totally odd. Suppose that $\overline{\rho}$ is (irreducible and) modular; and that it is unramified at every place of F above p. Then $\overline{\rho}$ arises from a mod p Hilbert modular eigenform of pallarel weight one (as defined above).

Remark

The easier direction– the mod p Galois representation associated to a mod p Hilbert modular eigenform of parallel weight one is unramified at every place of F above p– is known by Dimitrov-Wiese, Deo-D-W, and De Maria.

When $F = \mathbb{Q}$, this is a theorem of Gross when the characteristic polynomial of $\overline{\rho}(\mathrm{Frob}_p)$ has distinct roots, and of Coleman-Voloch when $p > 2$. Gee-Kassaei proves the Hilbert case of Gross' theorem when p is unramified in F.

A rough sketch of the proof

Step 1: Over a sufficiently large finite soluble totally real extension F' of F, any totally odd representation $\rho : \mathrm{Gal}(\overline{F}/F') \to \mathrm{GL}_2(\mathcal{O}_L)$, reducible at every place of F' above p, is pro-modular:

$$\begin{array}{ccc} R^\circ & \twoheadrightarrow & T^\circ \\ \downarrow & & \downarrow \\ R^\circ/I & \twoheadleftarrow & R^\circ/J \end{array}$$

where

- the universal ring for reducible-above-p deformations of $\overline{\rho}$ (together with Frobenius eigenvalues above p),
- the Hida ordinary Hecke algebra localised at $\overline{\rho}$,
- $J = \ker(R^\circ \twoheadrightarrow T^\circ)$,
- I is the ideal given by the trace $R^\circ \to \mathcal{O}_L$ of ρ.

Step 2: Prove that there are at least $2^{\{v|p\}}$ mod p Hilbert modular eigenforms of parallel weight p and of level p-old at p, whose associated Galois representation are all $\overline{\rho}$.

Step 3: Lift (linear combinations of) eigenforms

$$S_1(\Gamma, \mathbb{F}_L) \xrightarrow[Fr]{H} \cdots \xrightarrow[Fr]{H} S_p(\Gamma, \mathbb{F}_L)$$

recursively along:

- multiplication by powers H of Hasse invariants
- and partial Frobenius operators $Fr : \sum c_n q^n \mapsto \sum c_n q^{pn}$

while maintaining the multiplicity $\geq 2^{\{v|p\}}$ at all steps.

We make use of q-expansions and $\mathrm{Im}\, Fr = \ker \theta$ where
$\theta : \sum c_n q^n \mapsto \sum n c_n q^n$

Remark

This is a mod p analogue of the following theorem about the Fontaine-Mazur conjecture; but the proof for this mod p analogue is much harder!

Theorem (S)

Let $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathscr{O}_L)$ be a continuous representation. Suppose that

- ρ is totally odd,
- $\overline{\rho}$ is absolutely irreducible (if $p > 2$, then $\overline{\rho}$ is allowed to be reducible),
- $\overline{\rho}$ is p-ordinary modular,
- the image of the inertia subgroup at every place of F above p is finite (=the Hodge-Tate weights are all equal).

Then ρ arises from a Hilbert modular eigenform of parallel weight one. In particular, the image of ρ is finite, i.e., the Fontaine-Mazur conjecture holds for ρ.