Generalized special cycles and theta series

Yousheng Shi

Department of Mathematics
University of Wisconsin

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Classical theta series

- Define
$$\theta(\tau) = \sum_{d\in\mathbb{Z}} e^{2\pi i d^2 \tau} = \sum_{n\in\mathbb{Z}} r(n) e^{2\pi i n\tau},$$
where $r(n) = \sharp\{x \in \mathbb{Z} \mid x^2 = n\}$.
- Poisson summation formula:
$$\theta(-\frac{1}{4\tau}) = \sqrt{-2i\tau}\theta(\tau) \Rightarrow \theta(\frac{\tau}{4\tau+1}) = \sqrt{4\tau+1}\theta(\tau).$$
- $\theta(\tau)$ is a weight $\frac{1}{2}$ modular form for the congruence subgroup
$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \pmod 4 \right\} \subset \mathrm{SL}_2(\mathbb{Z}).$$

Classical theta series (continued)

- More generally, for a positive definite quadratic form Q with integer coefficients in n variables. Define

  $$r_Q(n) = \sharp\{x \in \mathbb{Z}^n \mid Q(x) = n\}.$$

  Then

  $$\theta_Q(\tau) = \sum_{d \in \mathbb{Z}^n} e^{2\pi i Q(d)\tau} = \sum_{n \in \mathbb{Z}} r_Q(n) e^{2\pi i n\tau}$$

  is a weight $\frac{n}{2}$ modular form for some congruence subgroup $\Gamma(N)$.
- For the quadratic form $Q(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2$, $\theta_Q(\tau) = \theta(\tau)^4$. $\theta_Q(\tau)$ is a weight 2 modular form. By computing its Fourier coefficients we get the classical theorem of Lagrange (for $n \geq 0$):

  $$r_Q(n) = 8\sigma(n) - 32\sigma(\frac{n}{4}),$$

  where $\sigma(n) = \sum_{d|n} d$.

Dual reductive pairs

- A dual reductive pair (in the sense of Howe) is a pair of subgroups (G, G') of the group $\mathrm{Sp}(\mathbb{W})$ of a vector space $\mathbb{W}$ with a symplectic form $\langle\langle , \rangle\rangle$ such that G is the centralizer of G' in $\mathrm{Sp}(\mathbb{W})$ and vice versa, and these groups acts reductively on $\mathbb{W}$.
- Example: Let V be a **k**-vector space with a symmetric form $(,)$, W be a **k**-vector space with a symplectic form $<, >$. Then $\langle\langle , \rangle\rangle = (,) \otimes_{\mathbf{k}} <, >$ is a symplectic form on $\mathbb{W} = V \otimes_{\mathbf{k}} W$. This gives the dual pair $(\mathrm{O}(V), \mathrm{Sp}(W))$.
- For a quadratic extension of fields k/k_0, we have unitary dual pairs $(\mathrm{U}(V), \mathrm{U}(W))$. Hermitian form $(,)$ on V and skew Hermitian form $<, >$ on W. $\langle\langle , \rangle\rangle = \mathrm{tr}_{k/k_0}(,) \otimes_{\mathbf{k}} <, >$ is a symplectic form on $\mathbb{W} = V \otimes_{\mathbf{k}} W$.

Re-interpretation of theta series

- The philosophy of Siegel, Weil and Howe: theta series of G is a modular form of G'.
- Example: $G = \mathrm{O}(V)$, $G' = \mathrm{SL}_2(\mathbb{Q})$. Then $\mathrm{SL}_2(\mathbb{A})$ acts on $\mathcal{S}(V(\mathbb{A}))$ by the Weil representation ω. Let φ be a Bruhat-Schwartz function on $V(\mathbb{A})$, then the theta function

$$\theta_\varphi(g') = \sum_{\mathbf{x} \in V} \omega(g')\varphi(x)$$

is $\mathrm{SL}_2(\mathbb{Q})$-invariant:

$$\theta_\varphi(\gamma g') = \theta_\varphi(g').$$

- Take $\varphi = \exp(-2\pi i Q(x)) \otimes \varphi(x)$ where φ is the characteristic function of $\mathbb{Z}^4 \otimes \mathbb{A}_f$.Then θ_φ is the Adelisation of θ_Q.

The geometry of indefinite forms

- Let $\mathbf{k}$ be an imaginary quadratic field. V is a Hermitian space over $\mathbf{k}$ with the form $(,)$ of signature (p,q) and an $\mathcal{O}_{\mathbf{k}}$-lattice $\mathcal{L}$. Let $G = \mathrm{U}(V,(,))$. For a congruence subgroup Γ of G fixing $\mathcal{L}$, we have the locally symmetric space (connected Shimura variety) $M = \Gamma \backslash D$, where

$$D = \{z \mid z \text{ is a negative } q\text{-plane in } V \otimes_{\mathbf{k}} \mathbb{C}\}$$

  is the symmetric space of $G(\mathbb{R})$.
- Can replace $G = \mathrm{U}(V,(,))$ by $G = \mathrm{GU}(V,(,))$ and work on the Shimura variety

$$\mathrm{Sh}(G,K) = G(\mathbf{k}) \backslash D \times G(\mathbb{A}_f)/K.$$

- For the rest of the talk let $G = G(\mathbb{R})$.

Special cycles

- For $\mathbf{x} \in V^r$ such that $(\mathbf{x}, \mathbf{x}) > 0$. Define

  $$U = \mathrm{span}_{\mathbb{C}}\{\mathbf{x}\}, G_U = \{g \in G \mid gx = x, \forall x \in U\} = G(U^{\perp}).$$

  Denote the symmetric space of G_U by D_U or $D(U^{\perp})$. Let $\Gamma_U = \Gamma \cap G_U$. Then $C_{\mathbf{x}} = C_U = \Gamma_U \backslash D_U$ is a subvariety of $\Gamma \backslash D$. We call C_U a special cycles. Intuitively, this just corresponds to $\mathrm{U}(p-r, q) \hookrightarrow \mathrm{U}(p, q)$.
- On the symmetric space level:

  $$\{\text{negative } q\text{-planes in } U^{\perp}\} \hookrightarrow \{\text{negative } q\text{-planes in } V\}.$$

Special cycles (continued)

- Let T be a rank r Hermitian matrix with values in $\mathcal{O}_{\mathbf{k}}$. Then $\{\mathrm{x} \in \mathcal{L}^r \mid (\mathrm{x}, \mathrm{x}) = T\}$ consists of a finite number of Γ orbits $\{\Gamma \cdot \mathrm{x}_1, \ldots \Gamma \cdot x_\ell\}$. Define

$$C_T = \sum_{j=1}^{\ell} C_{\mathrm{x}_j}.$$

- The cycle C_T has a moduli interpretation and can be defined on the integral model of the unitary Shimura variety.
- If V has signature $(1, 1)$, $r = 1$, and $\Gamma = \Gamma_0(N)$, C_m is (a union of) the image of the Heegner point under the m-th Heck operator on modular curve.

Geometric theta series

- Dual pair: $\mathrm{U}(p,q) \times \mathrm{U}(r,r)$.
- The symmetric space of $\mathrm{U}(r,r)$ is

$$\mathcal{H}_r = \{\tau = u + iv \mid u \in \mathrm{Herm}_r(\mathbb{C}), v \in \mathrm{Herm}_{r,>0}(\mathbb{C})\}.$$

- Theorem (Kudla-Millson 1980s): Assume Γ is co-compact. The geometric theta series

$$\theta(\tau) = \sum_{T \in \mathrm{Herm}_{r,\geq 0}(\mathcal{O}_{\mathbf{k}})} [C_T] \exp(2\pi i \mathrm{tr}(T\tau)) \in H_*(\Gamma \backslash D, \mathbb{C})$$

is a modular form on $\mathcal{H}_r$ of weight $\frac{p+q}{2}$ with respect to some congruence subgroup. The definition of C_T has to be modified if T is only semi-positive definite.

Remarks on KM theory

- The work of Kudla-Millson is inspired by the work of Hirzbruch-Zagier on Hilbert modular surfaces.
- Method: Construct differential forms with values in the Weil representation.
- One application (Bergeron-Millson-Moeglin): Cases of Hodge conjecture for $\mathrm{O}(n, 2)$ and $\mathrm{U}(n, 1)$ Shimura varieties.

Generalized special cycles

- What if $(\mathbf{x}, \mathbf{x})$ has signature (r, s) with $s > 0$?
- Let U be a subspace of V of signature (r, s) $(0 \leq r \leq p$, $0 \leq s \leq q)$. Then $G_U \cong \mathrm{U}(p-r, q-s)$ and D_U is the set of negative $(q-s)$-planes in $U^\perp$.
- Choose a point $z' \in D(U)$ which is a negative s-plane in U. Define an embedding $s_{z'} : D_U \to D$:

$$z \mapsto z' \oplus z.$$

  We call the image $D_{U,z'}$ (or $D_{\mathrm{x},z'}$). It is totally geodesic and complex analytic in D.

Generalized special cycles (continued)

- Let $\Gamma_U = \Gamma \cap G_U$. The embedding $s_{z'}$ decsends to a map $\Gamma_U \backslash D_{U,z'} \to M = \Gamma \backslash D$ which we still denote as $s_{z'}$. The image is an algebraic subvariety: a generalized special cycle denoted as $C_{U,z'}$. The homology class $[C_{U,z'}]$ does not depends on z'.
- In general $s_{z'}$ is not an embedding, however we can pass to subgroups of Γ such that the corresponding map is an embedding.
- When $G = \mathrm{U}(p,q)$, $\mathrm{O}(p,q)$ or $\mathrm{Sp}(p,q)$ and U is positive definite, the choice of z' in the above definition is unnecessary. The cycle C_U is the special cycle defined by Kudla-Millson.

Main result and strategy

- For $T \in \mathrm{Herm}_{r+s}(\mathcal{O}_{\mathbf{k}})$ with signature (r, s). Define C_T as in the case of special cycles.
- We would like to construct a theta series whose "non-degenerate" Fourier coefficients are C_T.
- Our strategy is similar to that of Kudla-Millson. The key step is to find a special cocycle φ in the relative Lie algebra cohomology with values in the Weil representation which will give rise to the Poincaré duals of generalized special cycles . Then we apply the theta distribution to φ to get $\theta(\varphi)$ which will be automatically automorphic.
- My work actually deals with the case $G = \mathrm{U}(p, q)$, $\mathrm{Sp}(2n, \mathbb{R})$ or $\mathrm{O}^*(2n)$.

The Weil representation and the theta correspondence

- The Weil representation ω: A certain double cover $\mathrm{Mp}(2n,\mathbb{R})$ of $\mathrm{Sp}(2n,\mathbb{R})$ acts on $L^2(\mathbb{R}^n)$ unitarily and on $\mathcal{S}(\mathbb{R}^n)$ smoothly. We call this representation the Weil representation (or the oscillator representation)
- The Schrödinger model $\mathcal{S}(\mathbb{R}^n)$ is convenient for studying geometry. The infinitesimal Fock model $\mathfrak{W} \cong \mathrm{Pol}(\mathbb{C}^n)$ is convenient for studying K-types.
- Dual reductive pair: $G \times G' \subset \mathrm{Sp}(2n,\mathbb{R})$ can be lifted to $\widetilde{G} \times \widetilde{G'} \subset \mathrm{Mp}(2n,\mathbb{R})$.
- Theta correspondence: one-to-one correspondence of representations of $\widetilde{G}$ and $\widetilde{G'}$ that "occurs" in $(\mathcal{S}(\mathbb{R}^n),\omega)$:

$$\pi \leftrightarrow \pi' = \theta_{G,G'}(\pi)$$

  where $\pi \in R(\widetilde{G},\omega), \pi' \in R(\widetilde{G'},\omega)$.

Lie algebra cohomology $H^\bullet(\mathfrak{g}, K; \mathfrak{W})$

- For each maximal compact subgroup K of G. Cartan decomposition:

$$\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} = \mathfrak{k} + \mathfrak{p}_- + \mathfrak{p}_+.$$

- Chain complex:

$$C^\bullet(\mathfrak{g}, K; \mathfrak{W}) = \mathrm{Hom}_K(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}), \mathfrak{W}) \cong (\wedge^\bullet(\mathfrak{p}^*) \otimes \mathfrak{W})^K,$$

  where $\mathfrak{g}$ the Lie algebra of G with the Cartan decomposition $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$. $\mathfrak{W}$ is the Schrödinger model or infinitesimal Fock model of the Weil representation.
- There is an isomorphism of chain complex:

$$C^\bullet(\mathfrak{g}, K; \mathfrak{W}) \xrightarrow{\sim} \Omega(D, \mathfrak{W})^G : \psi \mapsto \tilde{\psi}(g, x) \triangleq (L_{g^{-1}})^*(\psi)(g^{-1}x).$$

$$ev_x : \Omega(D, \mathfrak{W})^G \to \Omega(D, \mathbb{C})^{G_x}.$$

Anderson's thesis

- In his thesis, Greg Anderson constructed classes in the Dolbeault cohomology group $H^{\bullet,0}(\mathfrak{g}, \mathfrak{k}; \mathfrak{W})$.
- In case of the unitary group $\mathrm{U}(p, q)$, for a pair of integers $1 \leq r \leq p, 1 \leq s \leq q$, Anderson's thesis constructed a cocycle φ_+ in the group $H^{(R,0)}(\mathfrak{g}, \mathfrak{k}, \mathfrak{W}_+^{\mathfrak{p}-})$ where $\mathfrak{W}_+$ is the Weil representation of the compact dual pair $\mathrm{U}(p, q) \times \mathrm{U}(0, r + s)$ and $R = pq - (p - r)(q - s)$ which is the complex co-dimension of $C_{U,z'}$ if U has signature (r, s). $\mathfrak{W}_+^{\mathfrak{p}-}$ is the subspace of $\mathfrak{W}_+$ annihilated by $\mathfrak{p}^-$.
- His construction essentially realized a Vogan-Zuckerman representation $\mathcal{A}_{\mathfrak{q}}$ as a sub representation in $\mathfrak{W}$. As a result, one can show that φ_+ is closed ($d\varphi_+ = 0$).

Outer wedge product in the Fock model

- Let $\mathfrak{W}_+$ be the Polynomial Fock space of the dual pair $(\mathrm{U}(p,q), \mathrm{U}(0, r+s))$ and $\varphi_+ \in \mathrm{Hom}_K(\wedge^R \mathfrak{p}^+, \mathfrak{W}_+^{\mathfrak{p}^-})$ be the cocycly of Anderson.
- A mirror construction: $\varphi_- \in \mathrm{Hom}_K(\wedge^R \mathfrak{p}^-, \mathfrak{W}_-^{\mathfrak{p}^+})$. Here $\mathfrak{W}_-$ is the Polynomial Fock space of $(\mathrm{U}(p,q), \mathrm{U}(r+s, 0))$.
- Both $\mathfrak{W}_+$ and $\mathfrak{W}_-$ can be realized as Polynomial space on $\mathbb{C}^{r+s}$. $\mathfrak{W} = \mathfrak{W}_+ \boxtimes \mathfrak{W}_-$ is the Polynomial Fock space of $(\mathrm{U}(p,q), \mathrm{U}(r+s, r+s))$. Take the outer tensor product:

  $$\varphi = \varphi_+ \wedge \varphi_- \in \mathrm{Hom}_K(\wedge^{(R,R)} \mathfrak{p}, \mathfrak{W}).$$

  Since $d = d_+ \boxtimes 1 + 1 \boxtimes d_-$, $d\varphi = 0$.
- In the unitary case, when $s = 0$, φ is the cocycle construced by Kudla-Millson.

Theta distribution on φ

- Form the theta series using φ

$$\theta_\varphi(g, g') = \sum_{\mathbf{x} \in \mathcal{L}^{r+s}} \omega(g, g')\varphi(\mathbf{x}) = \sum_T \theta_{T,\varphi}(g, g'),$$

where

$$\theta_{T,\varphi}(g, g') = \sum_{\substack{\mathbf{x} \in \mathcal{L}^{r+s} \\ (\mathbf{x},\mathbf{x}) = T}} \omega(g, g')\varphi(x).$$

- By the theory of Weil, there is a congruence subgroup $\Gamma' \in G'$ such that θ_φ is Γ'-invariant.

Main theorem

- Let T be a rank $r+s$ Hermitian matrix of signature (r,s). Then $\{\mathrm{x} \in \mathrm{L}^k \mid (\mathrm{x},\mathrm{x}) = T\}$ consists of a finite number of Γ orbits $\{\Gamma \cdot \mathrm{x}_1, \dots \Gamma \cdot x_\ell\}$. Define

$$[C_T] = \sum_{j=1}^{\ell} [C_{\mathrm{x}_j, z_j'}].$$

- **Theorem** (Shi) Assume again that Γ is co-compact. For $T \in \mathrm{Herm}_{r+s}(\mathcal{O}_{\mathbf{k}})$ with signature (r,s), we have

$$[\theta_{T,\varphi}(z,g')] = \kappa(g',T)\mathrm{PD}([C_T]),$$

where $PD([C_T]) \in H^*(M)$ is the Poincaré dual of $[C_T]$. And for a proper choice of g', $\kappa(g',T) \neq 0$.

Poincaré duality and Thom form

- Let M be a compact manifold and C a closed submanifold, we say a form φ is the Poincaré dual of C if for any $\eta \in \Omega^\bullet(M), d\eta = 0$:

$$\int_M \eta \wedge \varphi = \int_C \eta.$$

- Let $E \to C$ be a vector bundle over a compact manifold, we say that φ is a Thom for of $E \to C$ if the support of φ is vertically compact and for any $\eta \in \Omega^\bullet(E), d\eta = 0$:

$$\int_E \eta \wedge \varphi = \int_C \eta.$$

- There are generalizations of compactly supported Thom forms. These are rapidly decreasing forms on E satisfying the same defining equation as compactly supported Thom forms.

Geometry of the tube

- Let $M = \Gamma \backslash D$ and $E = E_U = \Gamma_U \backslash D$. E_U is topologically a vector bundle over the generalized special cycle $D_{U,z'}$.
- The unfolding lemma:

$$\int_M \eta \wedge \sum_{y \in \Gamma \cdot \mathbf{x}} \varphi(y) = \int_{E_U} \eta \wedge \varphi(\mathbf{x}).$$

- In order to show that $\sum_{y \in \Gamma \cdot \mathbf{x}} \varphi(y)$ is the Poincaré dual of $C_{\mathbf{x},z'}$, it suffices to show that $\varphi(\mathbf{x})$ is a Thom form for $E_U \to \Gamma_U \backslash D_{U,z'}$.

Rapid decrease of Schwartz function valued forms on the tube E_U

- Let $z_0 \in D$ be the fixed base point corresponding to K. Recall that

$$C^{\bullet}(\mathfrak{g}, K; \mathfrak{W}) \xrightarrow{\sim} \Omega(D, \mathfrak{W})^G : \psi \mapsto \tilde{\psi}(g, \mathrm{x}) \triangleq (L_{g^{-1}})^*(\psi)(g^{-1}\mathrm{x}).$$

- Define $d(z, D_{U,z'})$ to be the Riemannian distance from z to the cycle $D_{U,z'}$.
  Theorem (Shi)
  Let $U = \mathrm{span}\{\mathrm{x}\}$, $z' \in D(U)$ and $z_0 \in D_{U,z'}$. For any $\psi \in C^{\bullet}(\mathfrak{g}, K; \mathfrak{W})$, for any positive number ρ, there is a constant C_ρ such that

$$||\tilde{\psi}(g, \mathrm{x})|| \leq C_\rho \exp\{-\rho \cdot d(gz_0, D_{U,z'})\}.$$

An asymptotic estimate

Recall that we have a fibration $\pi : E_U \to C_{U,z'}$. Let F be any fiber of π. Then

$$\kappa(g', T) = \int_F \tilde{\varphi}(z, g', \mathrm{x}).$$

This period integral is hard to compute in general due to the fact that F is not a symmetric space. However, as $\lambda \to \infty$, $\tilde{\varphi}(z, g', \lambda \mathrm{x})$ is more and more concentrated near the neighborhood of $C_{U,z'}$ and its major term is the volume form of F at the base point $z_0 \in C_{U,z'}$. Hence we apply the method of Laplace to compute the asymptotic value of $\kappa(g', \lambda T)$.

Table of dual reductive pairs over $\mathbb{R}$

- Type I:
  $(\mathrm{O}(p,q), \mathrm{Sp}(2n,\mathbb{R})) \subset \mathrm{Sp}(2n(p+q),\mathbb{R})$
  $(\mathrm{U}(p,q), \mathrm{U}(r,s)) \subset \mathrm{Sp}(2(p+q)(r+s),\mathbb{R})$
  $(\mathrm{Sp}(p,q), \mathrm{O}^*(2n,\mathbb{R})) \subset \mathrm{Sp}(4n(p+q),\mathbb{R})$
  $(\mathrm{O}(n,\mathbb{C}), \mathrm{Sp}(2m,\mathbb{C})) \subset \mathrm{Sp}(4mn,\mathbb{R})$
- Type II:
  $(GL(m,\mathbb{R}), \mathrm{GL}(n,\mathbb{R})) \subset \mathrm{GL}(2mn,\mathbb{R})$
  $(GL(m,\mathbb{C}), \mathrm{GL}(n,\mathbb{C})) \subset \mathrm{GL}(4mn,\mathbb{R})$
  $(GL(m,\mathbb{H}), \mathrm{GL}(n,\mathbb{H})) \subset \mathrm{GL}(8mn,\mathbb{R})$
- We focus on dual reductive pairs of type I. In this case, G is the linear isometry group of $(V, (\,,\,))$ where V is D-vector space with $D = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and $(\,,\,)$ is a non-degenerate Hermitian or skew Hermitian form on V.

The other two families of Hermitian symmetric domains

- Beside the unitary dual pairs, my work can also be carried out when $G = \mathrm{Sp}(2n, \mathbb{R})$ or $G = \mathrm{O}^*(2n)$. These covers all the cases when the associated D is Hermtian symmetric except for the case $\mathrm{O}(n, 2)$.
- Dual pairs: $(\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(2r, 2r))$ and $(\mathrm{O}^*(2n), \mathrm{Sp}(r, r))$.
- What is the same: The definition of generalized special cycles (algebraic cycles) and the main technique (Weil representation and Anderson's thesis).
- What is different:
  - There is no special cycle in the sense of Kudla and Millson.
  - The corresponding form $(,)$ has no signature, so the main theorem is true for all non-degenerate Fourier coefficients.

Other dual pairs

- Can also let $G = \mathrm{O}(p, q)$, $\mathrm{Sp}(p, q)$, $\mathrm{O}(n, \mathbb{C})$ or $\mathrm{Sp}(2n, \mathbb{C})$. Can define generalized special cycles on the associated locally symmetric spaces. Similar theorems can be obtained (work in preparation by Millson and Shi).

Thank you!